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ELEMENTARY STATICS

(FOR B.A. STUDENTS)

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BY

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Thoroughly Revised
THIRD EDITION

LAHORE
UNIVERSAL PUBLISHING COMPANY
1945

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PREFACE TO THE FIRST EDITION

This work has been written to provide a suitable course of reading in Statics for students of Indian Universities taking up Mathematics as an elective subject for their B. A. degree examination. An elementary knowledge of Analytical Geometry and Calculus has been assumed. But these instruments of Analysis have been very sparingly used except in the chapter on centre of gravity, where the method of Integral Calculus has been freely employed to obtain many interesting results. This will be of great indirect assistance to the student in mastering the process of integration regarded as summation. In many problems on the determination of the centre of gravity, the "Integral Calculus method" is the only suitable one, geometrical methods being long and tedious.

The theorem of the *parallelogram of forces*, which forms the basis of all discussion of statical principles, has been merely *assumed*, as the formal proof of the theorem from the "*principle of transmissibility of force*" involves the idea of incommensurability, which is rather puzzling to a beginner. The alternative procedure of deducing the theorem from Newton's definition of force and the *parallelogram of accelerations* could not be adopted, as this book is intended to be studied independently of Dynamics.

The aim of the authors has been to present to the student as clearly as possible the principles of Statics and these have been amply illustrated by means of solved examples appended to each chapter. As Examination Papers have been freely consulted in the selection of these typical examples, they will be of special help to the student. Some of the topics like the principle of virtual work and its application to machines, the law of passive resistance (*e.g.*, resistance offered by friction), and the subject of "Strings" will be found useful by the more ambitious student.

The authors acknowledge their indebtedness to standard text-books on Statics by Minchin, Routh, Loney, Todhunter, Greaves and others, from which they have received valuable assistance both in the matter of the text and examples. They will feel grateful for notification of errors and misprints that might have crept into the book.

AUTHORS.

EXTRACT FROM PREFACE TO THE SECOND EDITION

The demand for a second edition of the book has provided the authors with an opportunity for writing afresh certain portions of the book and amplifying others. A notable change has been made as regards the subject of Centre of Gravity. The analytical methods have been separated from geometrical ones, and put together in a separate chapter. This arrangement would give the teacher the option of taking up the methods of Integral Calculus given in Chapter VIII at a later stage, when he feels that the meaning of Integration as summation has been properly grasped by the students. Most of the examples in this chapter, can however be done by the geometrical method.

The more difficult examples and Articles have been marked with an asterisk and can be taken up at the revision stage, or omitted by the less ambitious student.

October 2, 1941,
LAHORE.

S. N. DAS GUPTA
G. L. GUPTA.

PREFACE TO THE THIRD EDITION

There has been an insistent demand for the third edition of the book for more than a year, and the Authors and the Publishers regret very much that owing to abnormal conditions created by the War they were unable to bring out this edition earlier. The time that has elapsed has, however, been utilized in incorporating in this edition quite a large number of improvements. While all the essential features of the previous editions have been retained, the book has been thoroughly revised with a view to present the subject matter in a simpler and more attractive form. Alternative proofs of important theorems have been introduced, and a large number of illustrative examples and exercises have been added, many of which have been selected from recent Examination Papers of the Punjab and Delhi Universities. The more difficult among these have either been solved and illustrated by diagrams, or have been made easier by the addition of necessary 'hints'.

While it is sincerely hoped that the interest of the student in the solution of problems will be excited by the devices adopted, it is necessary to emphasise that the student should tackle with difficulties as far as possible independently of the hints or solutions given, so that he may be able to keep the initiative with himself. He should, in fact, try if he can discover alternative methods of solution.

The present edition will be found to meet fully the requirements of the B. A. and B. Sc. pass-course students of Indian Universities, and it will at the same time, be of immense help to Honours students.

The chapter on machines which has been excluded from the Syllabus of the Delhi Uuiversity also has been omitted from the present edition, and the space thus saved has been utilized, as detailed above.

September 6, 1945,
LAHORE.

S. N. DAS GUPTA,
G. L. GUPTA.

catalogued

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CHAPTER I

INTRODUCTION

1. The laws which govern natural phenemena or the action of natural forces, have formed a subject of study since very early times. The philosophers who studied this subject in its early stages gave it the name of *Natural Philosophy*. Its branch that was studied first of all, in a scientific manner, is known as **Mechanics**, which deals with *the motion of bodies and the causes which produce it* (rest being a particular case of motion). On account of its simplicity, Mechanics is treated even to-day as introductory to other departments of Natural Philosophy or Physics, as the subject is now called.

Mechanics has *three main branches* :—

Kinematics, *Kinetics*, and *Statics*.

The first two are usually grouped under the name of *Dynamics* (signifying motion), as opposed to Statics.

Kinematics is that part of Dynamics which treats of motion without reference to mass and force. It is the geometrical science of pure motion.

Statics is that branch which deals with the equilibrium of bodies, and examines how this state obtains and is to be kept.

2. The principles of Mechanics must have taken a long time to evolve. Here, as in most other sciences, a great deal of experience was necessary before any theoretical study of the subject could be undertaken. This experience was gained, in the present case, from efforts made to satisfy the human need of building houses and bridges. Thus the invention of mechanical contrivances required for the purposes of engineering, was an important landmark in the history of the Science of Mechanics. These contrivances, which have been called **Machines**, retained their position of importance even in Theoretical Mechanics until quite recent times.

3. Among these "Machines," **Lever** is the most elementary one. The earliest investigations in Mechanics, which may really be called scientific, relate to *Statics*, and in

Statics they relate to Lever. Archimedes of Syracuse (287-212 B. C.) gives in his treatise on the subject a number of propositions on Lever and Centre of Gravity. The *Principle of Lever* thus formed the basis of Statics in the beginning and continued to do so for a long time.

4. It was Stevinus (1548-1620) who first investigated the Mechanics of the *Inclined Plane* (another Machine), and was led from there to the enunciation of the *principle of parallelogram of forces.* So the credit is due to Stevinus for first turning the attention from machines to a purely theoretical law which has since formed the basis of discussion in the Science of Statics.

5. The "parallelogram of forces" enunciates an *experimental law.* It can be demonstrated mathematically by means of Newton's Second Law of Motion and, for this reason, mathematicians sometimes prefer to treat Statics as a sequel to the laws of Dynamics.

It must be clearly understood that what we call "laws," or fundamental concepts, are themselves so many *assumptions* on which the superstructure of a theoretical science is built. These assumptions may or may not be "self-evident axioms or truths." But they are framed, on the one hand, as the result of observation, or experiment (or experience), or both, and are capable of verification, on the other hand, by experiments in the laboratory or by their application to natural phenomena.

The function of Mathematics is mainly to develop in logical sequence, the subject on the basis of the assumptions made, which include definitions in addition to such laws as have been above referred to.

6. Rigid Body and Matter.—In Statics the concepts of Space, Matter, and Force are fundamental. Space and Matter cannot be defined : we have an intuitive perception of each. Kant used to regard space as a 'pure intuition *a priori*,' but 'modern inquirers are inclined to regard space as a concept and, in addition, as a concept which has been derived from experience.'

Space is measured in terms of length. In the British system, *the unit of length is the foot.* It is generally admitted that Matter can be perceived by the senses and can be distinguished from other objects of sense, such as, colour, form and sound.

A **solid** *is a part of space*, and a **body** *is defined to be a limited portion of matter.*

By a geometrical solid is meant the space occupied or imagined to be occupied by a body. With this space the material qualities of the body have nothing to do. Thus the part of space occupied by a human body is a geometrical solid.

The term *rigid body* will be used to denote a figure the distance between any two points of which *always* remains constant. Such a figure will be of invariable size and shape. It may be merely a *geometrical solid* (or *configuration*) or a *physical entity*. In the latter case, it is composed of *Matter* which may be considered to be indestructible, something which fills the space enclosed by a geometrical three dimensional figure, and possesses certain other properties, some of which we shall state later on. As stated above, no formal definition of matter can be given which will convey a complete idea if we had not the previous experience of it.

When the idea of **Matter** is associated with a point, we get what is called a *particle*. In this case the whole quantity of matter is *supposed* concentrated at a point. A particle, therefore, occupies no space but has simply a position.

A particle may be considered as an infinitesimal portion of Matter ; a solid body is thus conceived to be an aggregation of particles (which are held together by their mutual attractions). In a perfectly rigid body the distance between any pair of particles remains unaltered (no matter what forces act on the body).

The terms "rigid body" and "particle" are only mathematical conceptions. Their exact equivalents do not exist in Nature though there may be near approximations. A particle of matter will be supposed to be a very small body, but possessing a sensible mass and capable of being acted upon by forces.

7. Mass. *The quantity of matter contained in a body is called its mass.* It does not depend upon the space it occupies.

A slab of lead will contain twice as much matter as another of half the volume, but it should be obvious that the mass of a body cannot be measured by its volume or bulk alone. If a certain quantity of cotton, for instance, could be compressed to occupy half its original volume, the material contained remains the same; only it now occupies half the space. In the same way, a certain volume of lead will contain a great deal more of mass than the same volume of cotton. Howsoever a given volume of cotton be compressed, it could not be reduced to a size so that it may contain an equal quantity of matter with lead of that size.

The mass of a body will then depend upon (1) the mass per *unit* volume of the substance, under given conditions, of which a body is composed, and (2) the volume it occupies.

We now put these conclusions in a formal manner :—

Def Density.—*The mass per unit volume of a substance, under given conditions, is called its density.* If mass, density, and volume of a substance be represented by M, ρ, and V respectively, we have $M=\rho V$, *i.e.*, **mass=density × volume.**

Unit of Mass.—In the English system of measurement, a **pound** is taken as the unit of mass. It is equivalent to the mass of a piece of platinum which is preserved as a standard in the Exchequer's office.

It should be noted, that for theoretical purposes, two particles are supposed to differ from each other only in the case of their having different masses.

8. Force. "Force" is not simply the "muscular effort" by which we push or pull bodies. When a stone is released and it falls to the ground, it does so under the action of a force which we call the gravitational force of the Earth. When a ball rolled on the ground with a certain velocity, comes to rest after some time, it does so under the action of the forces of resistance offered by the ground (called *friction*) and the air. Again, when a man pushes against a wall, and the wall doesn't give way, it is because the force of *push exerted by the man is balanced by the equal force of counter-action caused by the wall opposite to the action of the man.*

Force may then be defined in the following manner :

Force is that which changes or tends to change the state of rest, or of uniform motion of a body.

"Uniform motion" means that (1) the direction of motion remains the same, *i.e.* the body moves in a straight line, and (2) that the body goes over equal distances in equal intervals of time, however small those intervals may be.

The definition of Force conveys the idea that " matter " by *itself is incapable of changing the state in which it may* happen to be. If it is at rest, it will continue to be at rest, while if it is moving with a certain velocity in a certain direction, it will continue to move in that direction with the same velocity. This latter statement may appear to be contrary to common experience, but if it were possible to eliminate the action of all resistances, a body once set in motion would never stop. An approximation to the hypothetical conditions may, however, be obtained by rolling a hard dry ball of ice along a *smooth dry slab of ice*, in vacuum

We have here an important property of Matter known as the property of **Inertia.** The law which expresses it, is known as the **Law of Inertia**.

Thus we see that the effect of a force or forces acting on a body is to cause it to change its position or shape. The ability of the body to resist the change in position is called *Inertia* and the ability to resist the change in shape is called *Rigidity.*

In the above explanations, the efforts which push or pull bodies, the gravitational pull of the Earth, the resistances offered by the ground, or the air, and the counter-action of the wall are all instances of Force. We can **classify forces** then, according to the forms in which they may appear, as follows :—

(1) **Tension.**—When a body is pushed or pulled by means of a rod or a string, the force which acts through the intermediate agent (a rod or a string), is called the *force for Tension*.

(2) **Attraction and Repulsion.**—When a body is attracted towards another without the agency of any other tangible means, as in the case of the gravitational pull of the Earth or in the pull exerted by a magnet on pieces of iron, the force is called *the force of Attraction.* In the case of *repulsion*, the bodies tend to separate from each other.

(3) **Action and Reaction.**—The force occurring in this form is always due to the action of another body with which the given body may be in actual contact or not. It is explained by **Newton's Law of Action and Reaction**, which says *that every action is accompanied by an equal and opposite reaction* ; this implies that *forces always occur in pairs*.

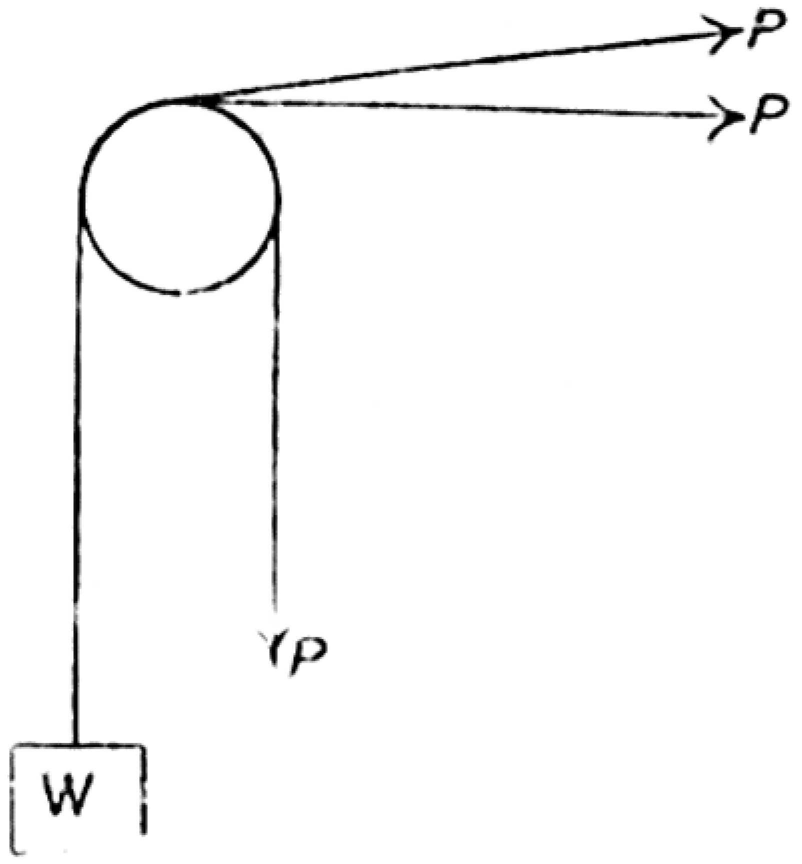
When one body presses against the other, the pressure is the *action*, and the counter-action which the second causes on the first is the *force of Reaction*. A man's hand pressing a wall, as explained above, does not go into the wall because the **action** *of the hand* and **reaction** *of the wall* are equal and opposite. Again, the force exerted by the Earth on as tone has its counterpart in the force exerted by the stone, on the Earth and if the Earth is not pulled towards the stone, it is only because the Earth has a considerably large mass than the stone. The action and reaction together constitute what is called a *stress*.

If the force exerted between two bodies in contact acts along their common normal, the bodies are said to be **smooth**; otherwise **rough.**

Note 1.—When tension is exerted by means of a *light inextensible string*, we infer from the following experiments that *tension is the same throughout its length*. By a light string we mean a string whose weight is inappreciable and can, therefore, be neglected.

1. A heavy weight W is tied to a *light inextensible string* and is kept in the vertical position by our hand. It is now found that whatever be the point at which the string is held, the same force (muscular effort) must be exerted to support the weight W and the tension at the point of the string is, thus, equal to W.

Again, if a *light inextensible string* passing over a *smooth peg* or *smooth grooved wheel* supports a weight W tied to one of its ends by means of a force P applied at the other end, it is found that the same force P would have to be exerted to support W, whatever be the inclination of the two portions of the string.



In these cases, at every point of the string,

$$\text{tension} = P = W.$$

Note 2. If the continuity of the string be interrupted by a knot, the string should be regarded as two distinct strings and tensions in these two portions are not necessarily equal. [See Art. 12].

Note 3. The following considerations will show that *tension is different at different points of an inextensible* **heavy string**.

Let a heavy inextensible string be suspended from the fixed point A, so that the string hangs vertically. Consider the equilibrium of a small portion of string PQ. It is held in equilibrium by the following forces :

(*i*) Tension T_1 acting *upwards* at P.

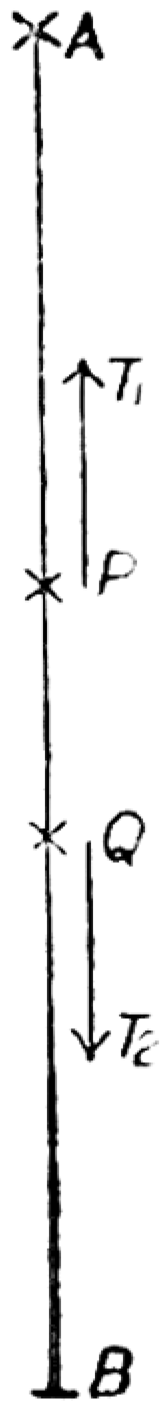
(*ii*) Tension T_2 acting *downwards* at Q.

(*iii*) The weight W of the portion PQ acting *downwards.*

For equilibrium, we must have

$$T_1 = T_2 + W.$$

Hence, $T_1 > T_2$, showing that the *tension of a string increases as we go upwards.*



Observe. (*a*) That Tension T_2 has two aspects ; as acting upon PQ, it is a *downward force*, but as acting upon the portion QB, it is an *upward force.* Similarly, T_1 is a *downward force* as acting on the fixed point A.

(*b*) If a weight W be attached at the lower extremity B, T_2 will have a greater value than before, as T_2 now supports the weight of the portion QB *plus* W, *i. e.* $T_2 = \text{wt. of } QB + W$.

Illustrative Example.—A light string hanging vertically from a fixed point supports a mass of 10 lbs. at its extremity and a mass of 15 lbs. is tied at its middle point ; find the tensions of the two portions of the string.

9. **Weight.**—*The force with which a body is attracted towards the centre of the Earth is called its weight.*

It is proved in books on Dynamics that the attraction of the Earth is really equivalent to the vertical force (which is usually called the weight) and the horizontal force brought into action due to the rotation of the Earth. Thus if the pull of the Earth is taken equivalent to the weight of a body, it is on the supposition that the effect due to rotation is neglected.

This explains a part of the reason why the force of gravity appears to vary from point to point on the earth's surface. For instance, a mass of a kilogramme at the equator will appear to act on a spring balance with a force equal only to the earth's attraction on 996½ grammes, because the weight of 3½ grammes goes towards producing the acceleration of the mass towards the earth's centre.

The student should guard himself against confusing weight with mass. In the first place the mass of a body is invariable, but its weight varies from place to place ; secondly, if a body could be taken to the centre of the earth or to a place in the interstellar space beyond the field of operation of the earth's attractive force, it will have no weight but there will be no change in its mass.

Unit of force.—*The weight of a unit of mass is usually taken in Statics as the unit of force.* Thus, in the British System, the weight of a pound is the unit of force and is called 1 pound weight, and is written as "1 lb wt."

In the French system, where the unit of mass is one gramme, the unit of force is "1 gm. wt."

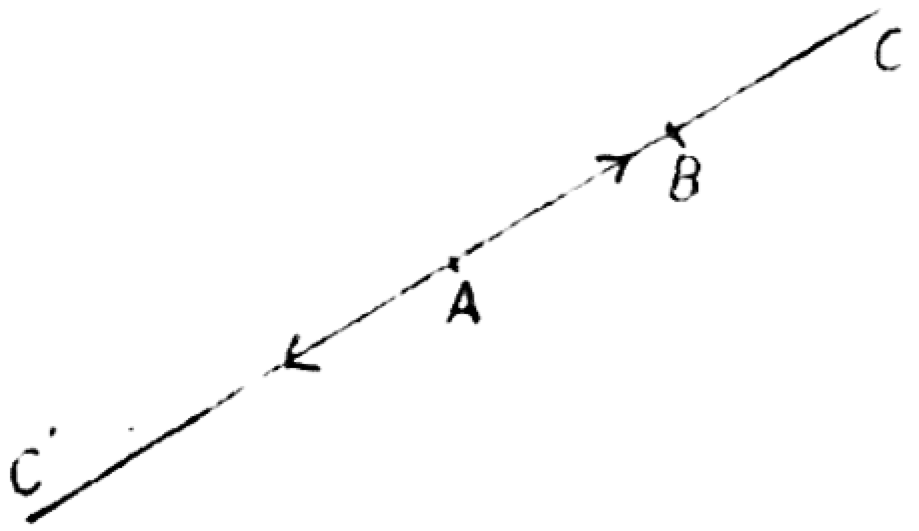
Since 1 gm. is a very small mass (453·6 gms.=1 lb. approximately), it is found more convenient to choose 1 kilogram-wt. as the unit of force.

If, for example, a mass of 15 lbs. rest on a horizontal table, it will exert a downward vertical force of 15 lbs. wt. (according to the above supposition) on the table and the reaction of the table will be equal to this force upwards. Similarly, if a mass of 15 lbs. were attached to a string, the pull exerted by the mass on the string will be taken to be 15 lbs. wt. vertically downwards.

10. Forces represented by straight lines.—Since a force is completely known when we know (1) *its point of application*, (2) *its direction*, (3) *its sense*, and (4) *its magnitude*, it can appropriately be represented by means of a straight line. *The point of application of a force together with its direction defines the* **line of action** *of the force, i. e.* the actual line in which the force acts.

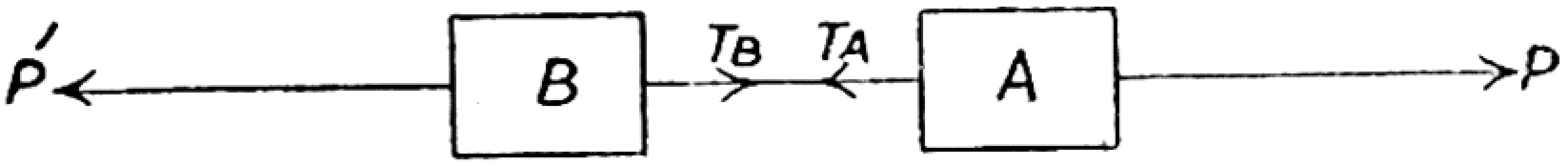
Thus if a force of 15 lbs. wt. acting at A along AB, be represented by the straight line AB, the arrow-head showing the direction and the sense, a force of 30 lbs. wt. acting at A along AB, will be represented by AC, where AC=2AB. If this second force were acting in the opposite direction, the arrow-head will be put in the opposite direction and the force represented by AC′, where AC′=AC in magnitude.

It may be noted that the magnitude of the force 15 lbs. is represented by the length of the straight line AB and its direction is determined by the angle which **AB** makes with some fixed line and its sense (forwards or backwards) by the arrowhead. Its direction and its sense may be regarded as equivalent to the direction in which it tends to make the point of application start.



11. External and Internal forces.—Forces (tensions, thrusts etc.) existing between parts of the *same body* are called *internal forces*. Force existing between a body and any body external to it is called an **external force**. These terms are *relative* ; a force which is internal in one investigation may be external in another. Whether a force is to be considered internal or external depends upon the body or system of bodies whose equilibrium or motion is the subject of investigation for the time being.

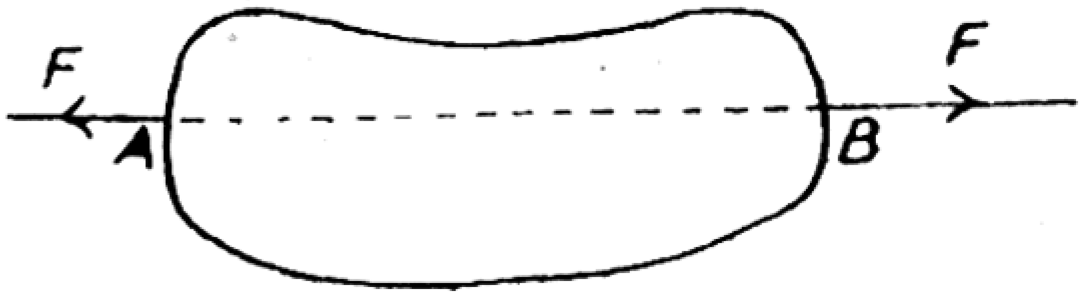
Let the bodies **A** and **B** attached by an inextensible string be kept in equilibrium by two opposing forces **P** and **P′**, as shown.



T_A and T_B are equal by Newton's 3rd law of motion. If we consider the equilibrium of both the bodies regarding A and B as a single body, then T_A and T_B are *internal forces*. If A and B are considered separate bodies, T_A and T_B are *external forces*. *Observe* that T_A and T_B are acting on two different bodies.

12. Principle of transmissibility of force.—On account of the property of rigidity in bodies, *a force can be transmitted from one point of a body to another lying on its line of action without any alteration in it.* This principle is of immense value in mechanics and brings about a great deal of simplicity.

When two equal and opposite forces, in the same straight line, act on a rigid body at points **A** and **B**, *the force F at B is transmitted to act at A* (by the principle assumed above).



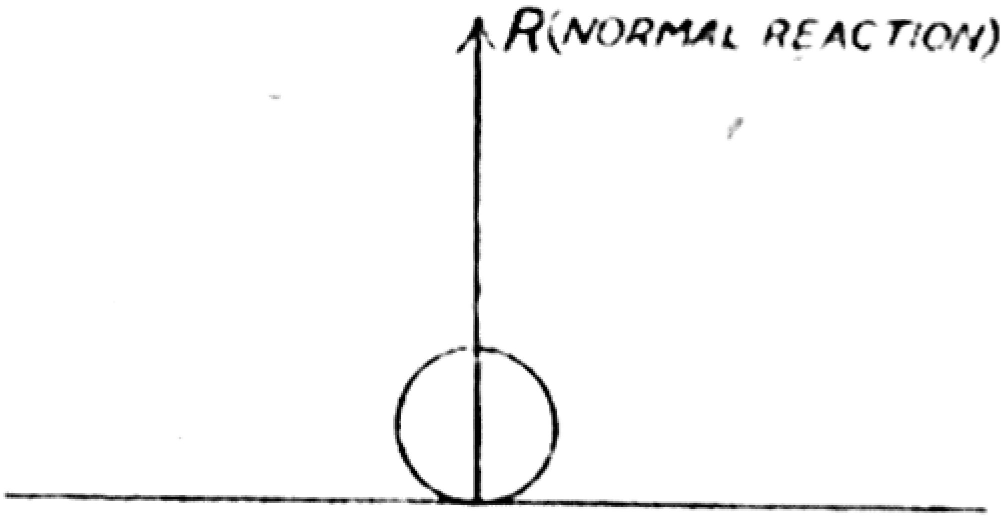
The body will thus be in equilibrium. If the forces F instead of pulling the body in opposite directions, compressed it, the resulting effect will be the same. But the *internal forces* called into play which would act on the particles composing the body would obviously be different in the two cases.

It is important to note that when a force acts on a **particle**, to know it fully, we must know (*i*) its *magnitude*, (*ii*) its *direction* and *sense*, and (*iii*) its *point of application*. But when a force acts on a **rigid body**, it is not necessary to know its point of application. It is sufficient if we know its *magnitude* and *line of action*, (The actual line in which the force acts).

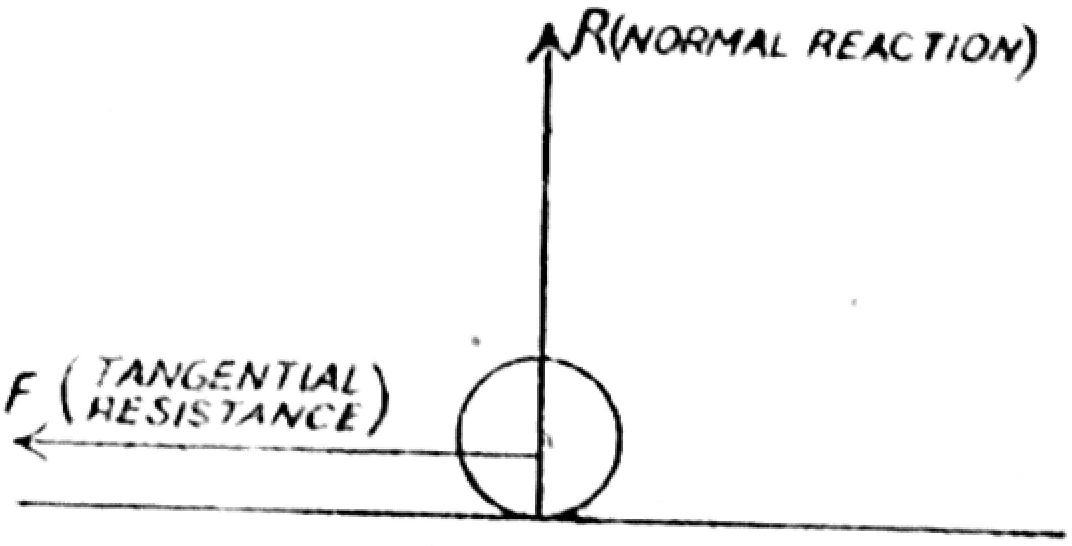
13. Smooth bodies. For the present we shall deal only with smooth bodies which have been defined in **Art. 8.** In such bodies there is no action, along (parallel to) the surface of contact whether they be in motion or in equilibrium. *The action (force) between two smooth bodies in contact is entirely perpendicular to the common surface at the point of contact.* For example, if a sphere rest on a horizontal plane *having a tendency to move to the right*, the reactions

(*i*) *when the bodies are smooth, and*

(*ii*) *when the bodies are rough* are as shown below.



Reaction between smooth bodies entirely normal : tangential resistance—0



Total reaction between rough bodies is the resultant of F and R.

Note.—Perfectly smooth bodies also have a hypothetical existence like perfectly rigid bodies, though in this case too we have in nature near approximations.

14. Translation and Rotation. We have to distinguish between *two kinds of motion* of a rigid body. When different points of the body move in parallel directions and receive equal displacements, the motion is said to be a *motion of translation.* If the body turns about on an axis either situated in the body or outside it, or if it rotates about a fixed point, the motion is said to be *a motion of rotation.*

15. Scalars and Vectors. There are certain quantities which are completely known when we know their *magnitudes only.* These are called *Scalar quantities.* But there are others which are not fully defined until we know their *magnitudes* and *directions* (including *senses*) both.

A quantity which is completely defined by its *magnitude, direction* and *sense* is called a **Vector quantity**. Velocity, acceleration, force and momentum are examples of Vector quantities. A quantity having only *magnitude* but no *direction* is called a **Scalar quantity.** Length, time, mass and energy are examples of Scalar quantities.

A vector has *not* necessarily a definite position in space.

A vector when it has a definite position is called a **Rotor** (*a localised vector*). Force is a *rotor.*

Scalar quantities of the same kind such as two masses or two lengths are combined by arithmetical addition. *Vector quantities* of the same kind are combined by the *parallelogram law* (*law of vector addition*).

If AB represents a vector, it is usual to *place* a bar over it and to write it as $\overline{AB}$, indicating that the order of the letters represents the direction and sense of the quantity.

The sign "+" will also be used to indicate vector addition. Thus, the parallelogram of forces (Art. 16), in vector notation, may be expressed by the equation

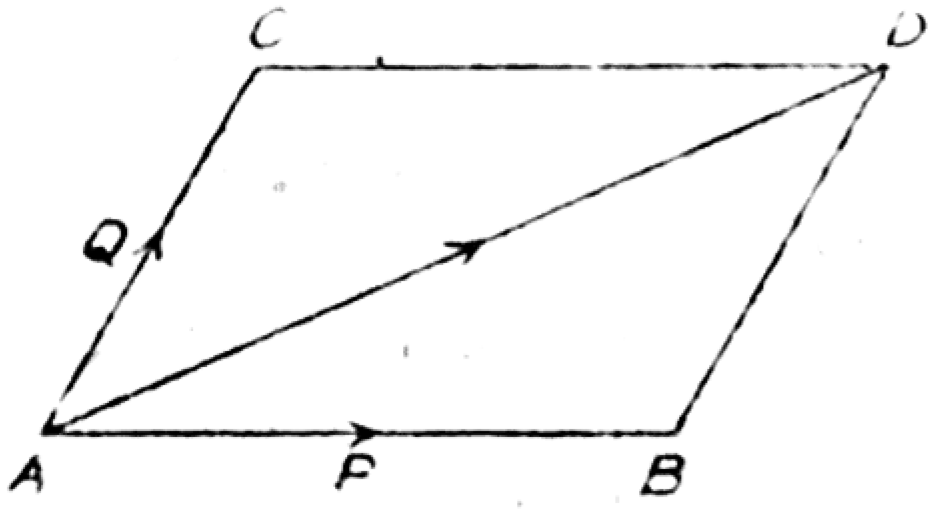
$$\overline{AB}+\overline{AC}=\overline{AD}.$$

CHAPTER II

COMPOSITION AND RESOLUTION OF FORCES ON A PARTICLE

16. Parallelogram of Forces.—When two forces act on a particle, their total effect is always equivalent to a single force (it may vanish as a special case), which is called their *resultant.* This resultant force is obtained by the *Law of the Parallelogram of Forces* which is stated as follows :—

If two forces acting on a particle be represented in magnitude and direction by the two sides of a parallelogram drawn through their point of application, the resultant will be represented by the diagonal passing through the point.



Let P and Q be two forces acting on a particle at A and represented, respectively, by the straight lines AB and AC. Then if the parallelogram ABDC be completed, the resultant force R will be represented by the diagonal AD.

The formal or mathematical proof of the theorem of the parallelogram of forces will be found in any book on Dynamics. The truth of the theorem will, however, be taken for granted in this book (Art. 5). It may be verified by means of experiments which can be easily arranged in the laboratory.

17. To find the magnitude and the direction of the resultant of two forces acting at a given angle.

Let the two given forces P and Q represented by AB and AC, act at an angle α.

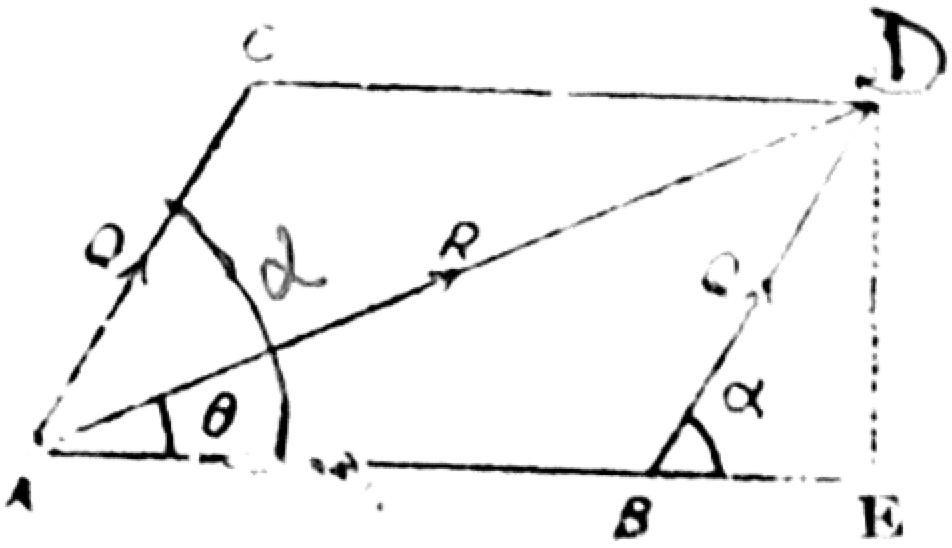


Fig. 1.

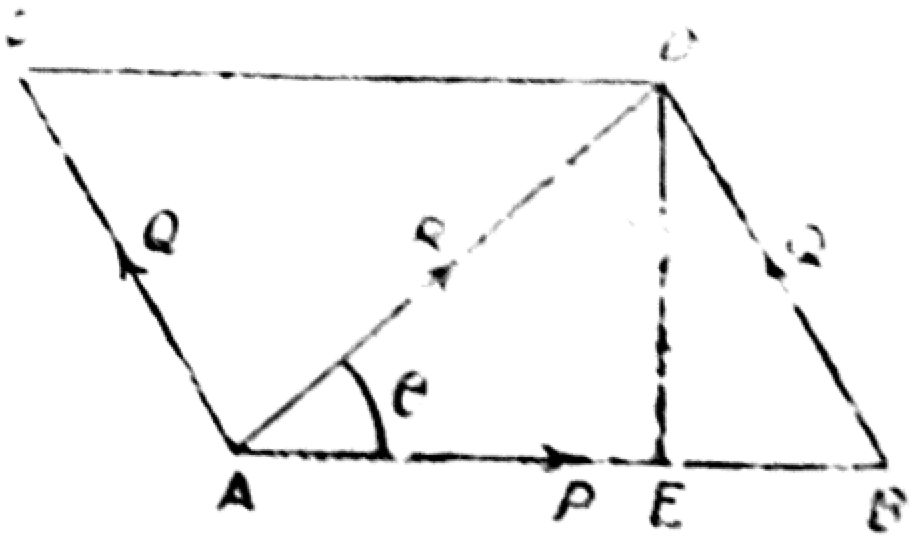


Fig 2.

Then in the parallelogram AD, angle $ABD=\pi-\alpha$, and hence

$$AD^2=AB^2+BD^2-2AB.\ BD\cos(\pi-\alpha)$$
$$=AB^2+AD^2+2AB.\ BD\cos\alpha.$$

Thus the magnitude R of the resultant is given by the formula

$$R^2=P^2+Q^2+2PQ\cos\alpha \qquad \ldots\ldots(1).$$

If θ be the angle which the resultant R makes with the direction of P, then drawing the perpendicular DE on AB or AB produced, we see that

$$\tan\theta=\frac{DE}{AE}=\frac{DE}{AB+BE}$$

(BE is +ve. in fig. 1 where α is an acute angle, and −ve in fig. 2, where α is obtuse)

$$=\frac{BD\sin\alpha}{AB+BD\cos\alpha}, \text{ (in both the cases).}$$

Hence $\tan\theta=\dfrac{Q\sin\alpha}{P+Q\cos\alpha}$,

$$\text{or}\quad \theta=\tan^{-1}\frac{Q\sin\alpha}{P+Q\cos\alpha} \qquad \ldots\ldots(2).$$

Cor. 1. If $\alpha=90°$, the formulae (1) and (2) become, respectively,

$$R^2=P^2+Q^2, \text{ and } \theta=\tan^{-1}\frac{Q}{P} \qquad \ldots\ldots(3).$$

Cor. 2. Again, if $P=Q$, $R^2=2P^2(1+\cos\alpha)$

$$=4P^2\cos^2\frac{\alpha}{2}$$

$$\text{or}\qquad R=2P\cos\frac{\alpha}{2} \qquad \ldots\ldots(4).$$

Also in this case $\tan\theta=\dfrac{Q\sin\alpha}{Q(1+\cos\alpha)}$

$$=\frac{2\sin\frac{\alpha}{2}\text{cas}\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}}=\tan\frac{\alpha}{2},$$

$$\theta=\frac{\alpha}{2} \qquad \ldots\ldots(5).$$

This last result is obvious otherwise also; there is no reason why R should incline more towards P than towards Q, when $P=Q$. Hence the *resultant of two equal forces bisects the angle at which they act, and its magnitude is equal to twice the force multiplied by the cosine of half the angle between the forces.*

Ex. Find geometrically the length of the diagonal of the parallelogram when the forces are equal, and obtain the formula (4).

Cor. 3. When $\alpha=0$, $\cos\alpha$ is maximum, and $R^2=(P+Q)^2$ or, $R=P+Q$; *i.e.* the resultant of two forces is maximum when they act in the same direction.

Cor. 4. When $\alpha=\pi$, $\cos\alpha$ is minimum, and $R=P-Q$, which shows that the resultant of two forces is minimum when they act in opposite directions.

It follows that *two forces can balance only when they are equal and opposite.*

Ex. 1. *At what angle must two equal forces act so that their resultant my be equal to each of them?*

Let α be the angle between them; then from the formula $R^3=P^2+Q^2+2PQ\cos\alpha$ we have, since $P=Q=R$,
$P^2=P^2+P^2+2P^2\cos\alpha=2P^3(1+\cos\alpha)$.

Hence $1+\cos\alpha=\frac{1}{2}$ or $\cos\alpha=-\frac{1}{2}$ $\therefore \alpha=120°$.

Otherwise.—The triangle ABD in fig. 2, art. **17**, will become equilateral.

Note.—Geometrical proofs will in certain cases, be found easier, and interesting.

Ex. 2. *Find the magnitudes of two forces such that if they act at right angles, their resultant is* $\sqrt{14}$ *lbs. weight, while if they act at an angle 120°, their resultant is* $\sqrt{13}$ *lbs. wt.*

By the question

(1) $P^2+Q^2=14$ (2) $P^3+Q^2+2PQ\cos 120°=13$ $(\cos 120°=-\frac{1}{2})$

Subtracting we get, $PQ=14-13=1$

$\therefore P+Q=\sqrt{P^2+Q^3+2PQ}=\sqrt{14+2}=4$

Similarly, $P-Q=\sqrt{P^2+Q^2-2PQ}=\sqrt{14-2}=2\sqrt{3}$

Hence, $P=2+\sqrt{3}$ lbs. wt.

and $Q=2-\sqrt{3}$ lbs. wt.

Exercises I (A)

1. Find.

(*i*) R, if $P=5$, $Q=11$, $\alpha=60^\circ$

(*ii*) α, if $P=12$, $Q=5$, $R=13$.

(*iii*) α, if $P=a^2-b^2$, $Q=2ab$, $R=a^2+b^2$.

(*iv*) α, if $P=2$, $Q=3$, $R=\sqrt{19}$.

(*v*) P and Q, if $P+Q=60$, $R=55$, $\alpha=60^\circ$

(*vi*) $\frac{P}{Q}$, if $\alpha=150^\circ$, R=smaller component.

2. The resultant of two forces at right angles is represented by 20 and it makes an angle of 30° with one of them. Find the forces.

3. Resultant of two equal forces acting at angle 60° is equal to R. Find the cosine of the angle between them when the resultant is $\frac{R}{2}$.

4. Two equal forces inclined at an angle of 60° act on a particle ; if their resultant is 12 lbs. find the forces.

5. The magnitudes of two forces are as 4 : 5 ; the direction of their resultant is at right angles to the smaller force. Find the ratio of the larger force to the resultant.

6. Forces of 40 lbs. and 24 lbs. have a resultant R and the resultant is inclined at 90° to the 24 lbs. force. Find R.

7. If the directions of two forces be inclined to each other at an angle of 135° find the ratio of their magnitudes in order that their resultant may be equal to the smaller force.

8. At what angle must the forces $P+Q$ and $P-Q$ act in order that their resultant may be $\sqrt{P^2+3Q^2}$?

9. The resultant of two forces is the same when their directions are inclined at an angle θ as when they are inclined at $\left(\frac{\pi}{4}-\theta\right)$. Show that $\tan\theta=\sqrt{2}-1$.

10. When two equal forces are inclined at an angle 2α their resultant is twice as great as when they are inclined at an angle 2β, show that $\cos\alpha=2\cos\beta$.

18. Equilibrium. *A number of forces acting on a body are said to have a* **resultant** *when their combined effect on the body is equivalent to the effect of a single force.* The process of obtaining the resultant is called the *composition of forces.* But *if the forces act in such a way that their total effect on the body is nil, the forces are said to be in equilibrium.* Such a system of forces does not, by the definition of Force, produce any change in the state of rest or of uniform motion of the body. If the body were initially at rest, it would remain at rest under the action of a system of forces in equilibrium.

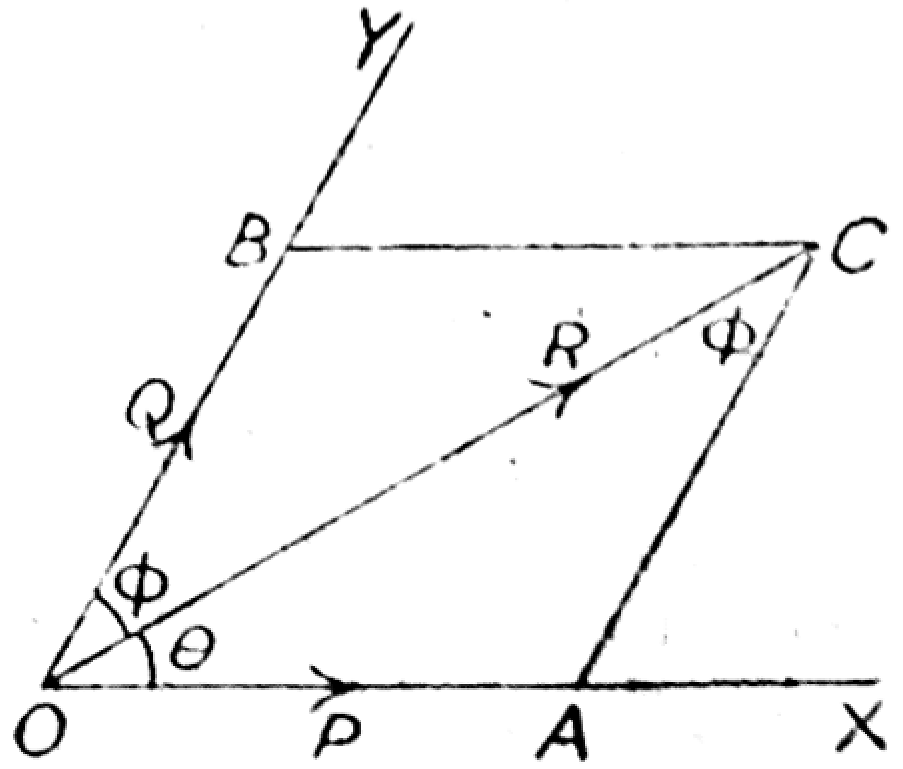
It has already been pointed out that a force cannot occur by itself without exciting other forces. External forces acting on a body call into play forces known as passive or internal forces. The resistance to change of position is inherent in a body and is of the nature of a passive force. In Statics we consider the cases in which the external forces and the internal forces form a system in equilibrium.

When more than two forces act on a body, it may be possible to compound them into one single force by successive application of the parallelogram law.

19. Resolution of Forces The converse of the "Composition of Forces" would be that a force being given we have to find the **component forces** in given directions, so that they may together be equivalent to the given force. This process is called "Resolution of Forces".

Components in assigned directions. *To find the component of a given force in two given directions in the same plane.*

Let OC represent the given force R, and let θ and φ be the angles which R makes with the given directions **OX** and **OY**.



Through C draw CA, CB parallel to OY and OX respectively; OACB is thus a parallelogram.

The components of R in the given directions are OA and OB (=*AC*).

From △ OAC, we have

$$\frac{OA}{\text{Sin } OCA}=\frac{AC}{\text{Sin } AOC}=\frac{OC}{\text{Sin } OAC}$$

i e.
$$\frac{OA}{\text{Sin } \phi}=\frac{OB}{\text{Sin } \theta}=\frac{OC}{\text{Sin } [180^\circ-(\theta+\phi)]}=\frac{R}{\text{Sin } (\theta+\phi)}$$

$$OA = R\frac{\text{Sin } \phi}{\text{Sin } (\theta+\phi)}$$

$$OB = R\frac{\text{Sin } \theta}{\text{Sin } (\theta+\phi)}$$

Note 1. If lengths OA, OB represent the forces P,Q respectively, then P and Q, the *components of R in the given directions OA and OB are given by*

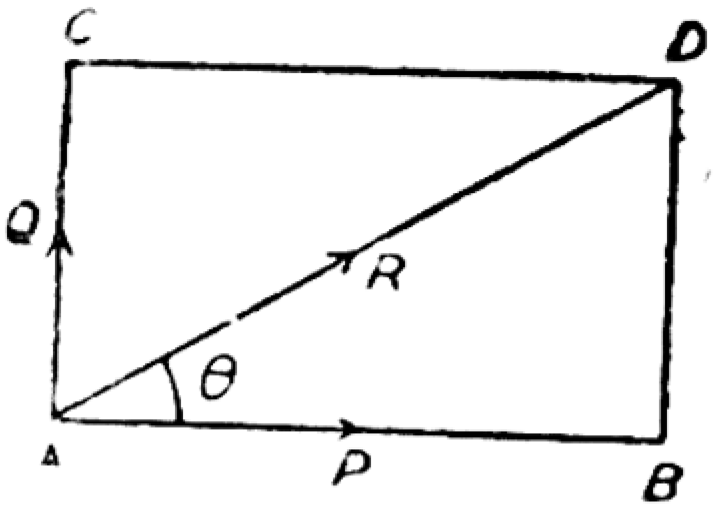
$$P = R\frac{\text{Sin } \phi}{\text{Sin } (\theta+\phi)}$$

$$Q = R\frac{\text{Sin } \theta}{\text{Sin } (\theta+\phi)}$$

Note 2. A force may be resolved into two component forces in an infinite number of ways, for OC may be the diagonal of an infinite number of parallelograms. The components will, however, be unique if θ and ϕ the directions of the components are given.

Note 3. In decomposing a force into two others, we can not speak of one component as quite independent of the other because the *magnitude* of OA will depend on the *directions* of OB. This is evident from the geometrical property of the parallelogram. Hence when we speak of the component of a force in any given direction we must know the direction of the other component. In practice, it is advantageous to take the second direction at right angles to the first.

20. Resolved Part. The special case when a given force is resolved in two directions at right angles to each other, is very important. *The component of a given force in a given direction so that the other component may be at right angles to the first component is called* the **Resolved part** *of the given force in the given direction.*



Thus R being the given force in the figure, its resolved part along AB is P, such that when P is compounded with a force at right angles to P, the resultant may be R.

We have directly from triangle ABD,

$$P = R \cos \theta.$$

i.e., the *resolved part of a force in a direction making an angle* θ *with it, is obtained by multiplying the force by cosine of the angle.*

The other component $Q = R \sin \theta$.

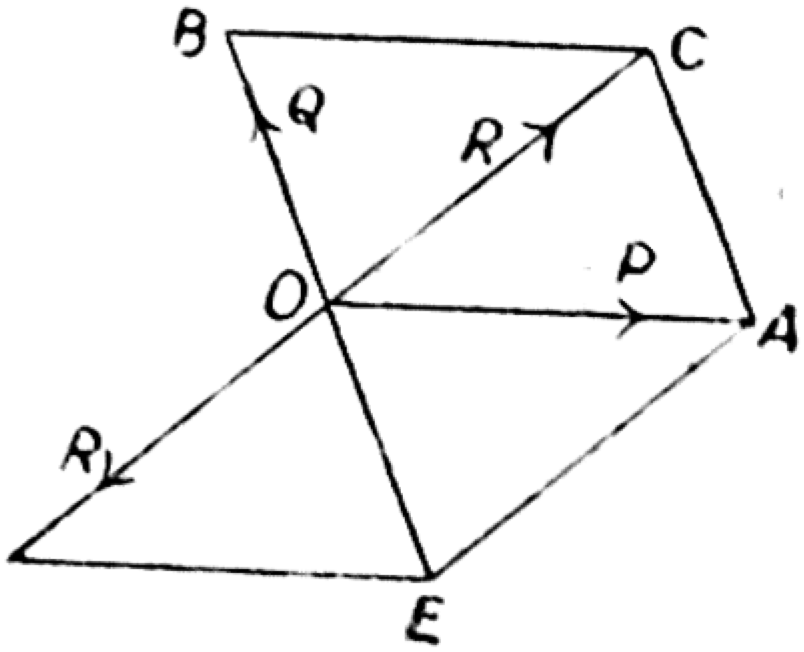
Geometrically, the resolved part of a force R represented by AB in a given direction is the orthogonal projection of AD in that direction.

Since the resolved part of a force in its perpendicular direction is zero, a force cannot produce any effect in the direction perpendicular to its line of action. It is for this reason that if a truck standing on the rails of a railway line be pushed perpendicular to the rails, it does not move along the line at all.

21. Duchayla's Proof.

Assuming that the diagonal represents the line of action of the resultant, to prove that the diagonal also represents the resultant in magnitude.

Let forces P and Q be represented by the lines OA and OB. Complete the parallelogram OACB. Then OC is *the line of action of the resultant.* Take a force R to balance the resultant of P and Q and let R be represented in magnitude and direction by OD.



Since the 3 forces P, Q, R are in equilibrium each of them is equal and opposite to the resultant of the other two.

Complete the $\|^{m}$ ODEA. Then OE is the direction of the resultant of P and R (from *data*). But the direction of P and R must be opposite to Q since the forces are in equilibrium. Hence OB and OE are in the same straight line.

$\therefore$ OEAC is a $\|^{m}$

Hence $EA = OC$.

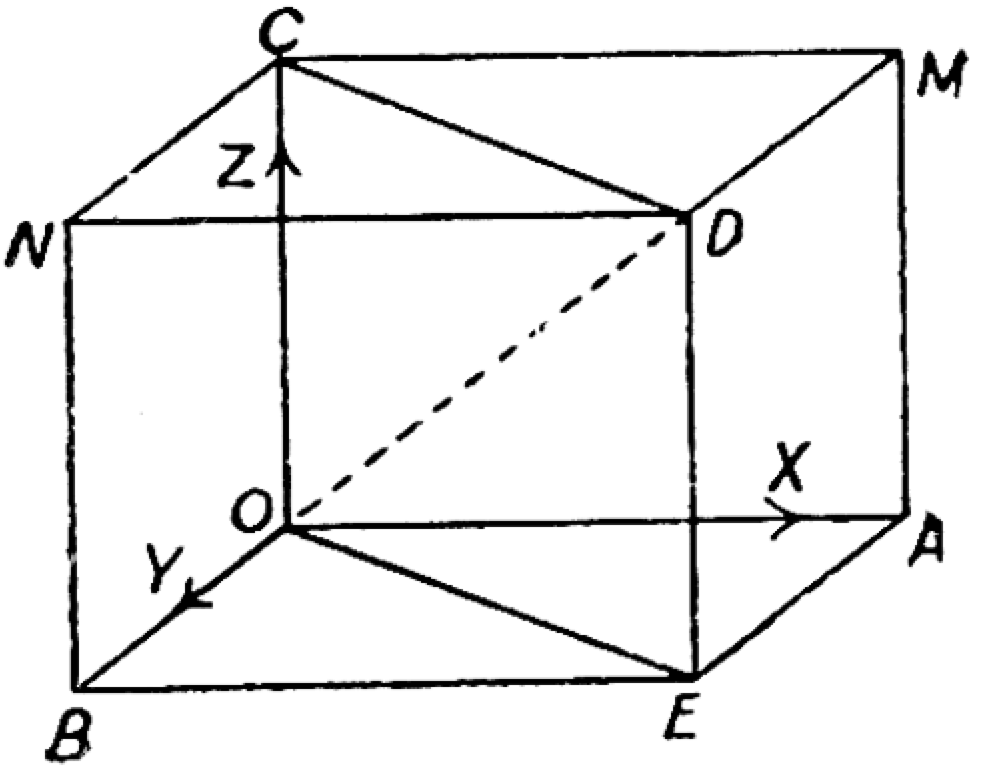
$\therefore \quad OC = OD$.

∴ OC represents the magnitude as well as the line of action of the resultant.

This is known as **Duchayla's Proof.**

22. Parallelopiped of Forces.

If three forces acting at a point are represented in magnitude and direction by the three edges of a parallelopiped passing through the point, their resultant is represented in magnitude and direction by the diagonal which passes through the point.



Resultant of X and Y (OA and OB) is represented by OE.

The resultant of OE and Z (OC) is represented by OD.

∴ OD represents the resultant of X, Y and Z.

Note. If the parallelopiped be rectangular

$$OE^2 = OA^2 + OB^2$$
$$OD^2 = OE^2 + OC^2$$
$$= OA^2 + OB^2 + OC^2$$
$$\therefore R^2 = X^2 + Y^2 + Z^2$$

Exercises I (B)

1. Two forces of 8 and 6 lbs wt. act at the angle of (*i*) 60°, (*ii*) 120°; find the magnitude and direction of the resultant in each case.

2. The resultant of two forces acting at right angles is 25 lbs. wt. and one of the components is 24 lbs. wt. find the direction and magnitude of the other component.

3. The resultant of two forces acting at the angle of 45° is $5\sqrt{5}$ lbs. wt. and one of the components is 5 lbs. wt. find the other component. Explain the double result.

4. Show that the resultant of two given forces increases as the angle between them is diminished. Hence find when the resultant will be maximum, and when minimum.

5. If the greatest and least value of the resultant of two forces be 40 lbs. wt. and 25 lbs. wt. find the forces.

6. R is the resultant of two forces P and Q acting at an angle α : if P be doubled, the resultant is doubled.

Show that $\alpha = \sin^{-1} \dfrac{\sqrt{16P^2 - 9Q^2}}{4P}$.

7. If four equal co-planer forces acting on a particle are in equilibrium show that they cancel each other in pairs.

8. Two equal forces act at a certain angle on a particle and have a certain resultant; also if the direction of one of the forces be reversed and its magnitude doubled, the resultant is of the same magnitude as before. Find the angle between the equal forces.

9. If two forces P and Q have a resultant R which is at right angles to P, show that Q is greater than P as well as R, while P may be greater or less than R.

10. If two forces acting at a given angle have a certain resultant, show that if the forces be interchanged, the magnitude of the resultant remains unchanged. Find the condition in order that the direction also may remain the same.

11. Forces P and Q acting at right angles have a resultant R. If P be increased by 2 lbs. and Q by 12 lbs., R also increases by 12 lbs. and the new resultant is the sum of Q and R. Find P and Q.

12. Two forces, one of which is double the other, have a certain resultant such that if the greater force be doubled again, and the smaller be increased by 4 lbs. wt., the resultant has the same direction as before. Find the forces and also find how the resultant is altered.

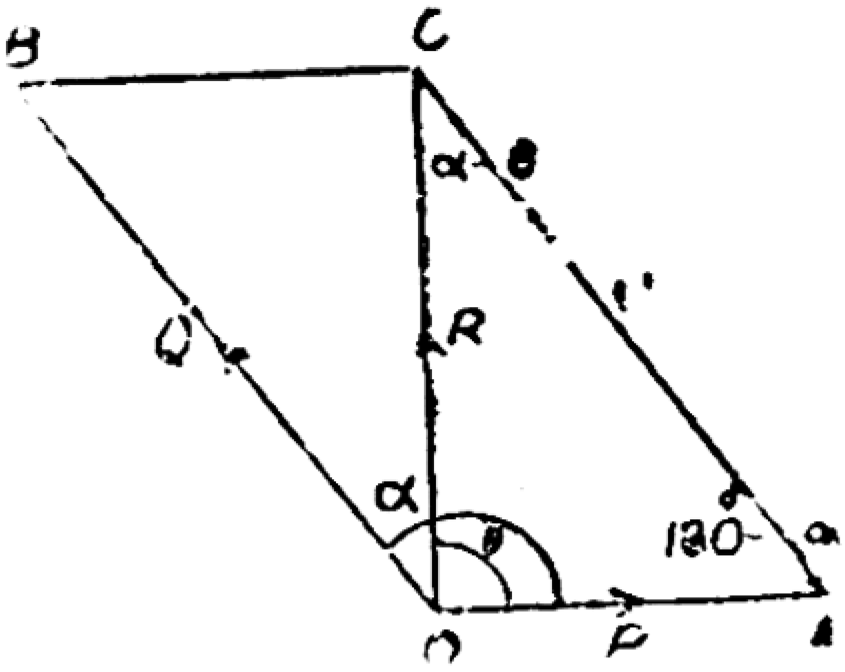
13. A given force R is resolved into two components so that one of the components at right angles to the given force is equal in magnitude to R itself; find the other component and its direction.

14. Resolve a given force R into two components making angles of 80° and 75° with it.

15. Two forces are in the ratio $2 : \sqrt{3}$; if their resultant be perpendicular to the smaller force, show that the resultant is half the greater force and is inclined at 60° to it.

16. If the direction and magnitude of the resultant of two forces, the directions of which are inclined at a given obtuse angle, be known, prove that one of the forces is greatest when the other is perpendicular to the resultant.

(P. U. 1914.)



Sol. Let the given force R make angle θ with one of the components P, and let the given angle between the components P and Q (suppose) be α. It should be noted that α is given while θ is to be determined. Completing the parallelogram, we have

$$\frac{Q}{\sin\theta}=\frac{P}{\sin(\alpha-\theta)}=\frac{R}{\sin\alpha}\qquad\text{so that } Q=\frac{R\sin\theta}{\sin\alpha}.$$

Since R and α are known, the value of Q depends upon the value of θ, and is obviously greatest when $\theta=90°$. The value of P is then $-R\cot\alpha$.

17. Find the resultant of six equal forces acting on a particle such that each may be inclined to the next at 30°.

18. ABCDEF is a regular hexagon and a force R acts along the diagonal AD. If this force be resolved into four equal forces acting along AB, AC, AE and AF, find the component forces.

19. The resultant of two forces is 2 P; the magnitude of one is P and the direction of the other makes an angle of 30° with the direction of the resultant. find the direction of the former and the magnitude of the latter.

20. If the greatest possible resultant of two forces P and Q is n times that of the least possible, show that their inclination when their resultant is half their sum, α is equal $\cos^{-1}\left(-\frac{n^2+2}{2(n^2-1)}\right)$.

21. Two forces P and Q include an angle θ and their resultant is R; if the angle is $\frac{\pi}{3}-\theta$, the resultant is $\frac{R}{2}$; if the angle is $\frac{\pi}{3}+\theta$, the resultant is $\frac{R}{3}$, show that

$$\cot\theta=\frac{59\sqrt{3}}{5}.$$

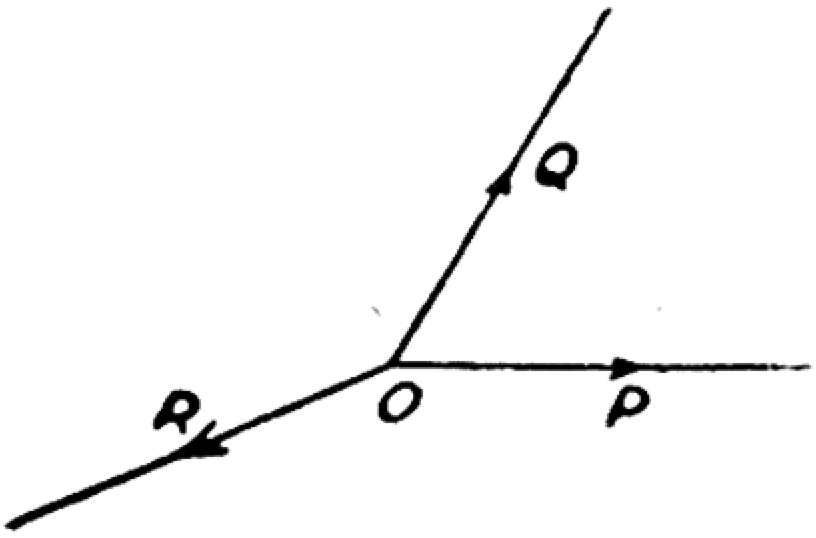
22. The magnitude and line of action of a force are given. Show how to decompose it into two *equal* components passing through two given points *A and B.*

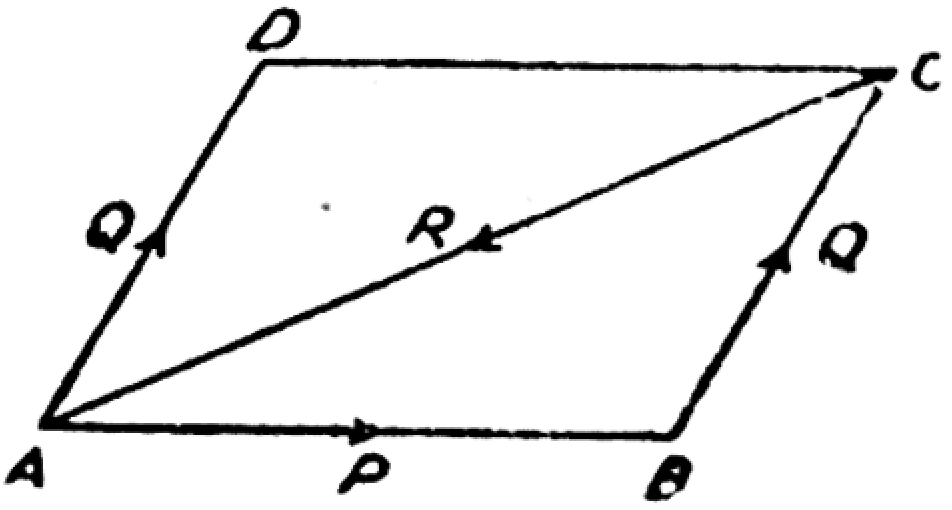
Hint.—From A draw AK perpendicular to the line of action KP of the force; produce AK to L, making KL=AK. Join LB cutting KP in P. PA, PB are the lines of action of the equal forces.

CHAPTER III

STATICS OF A PARTICLE

23. **Triangle of Forces.**—*If three forces acting on a particle be represented in magnitude and direction by the sides of a triangle, taken in order, they are in equilibrium.*





Let three forces P, Q, R acting on a particle at O, be represented in magnitude and direction by the sides AB, BC, CA, respectivly of the triangle ABC, taken in order; *then the three forces will be in equilibrium.*

Complete the parallelogram ABCD.

Since AD is equal and parallel to BC, Q may also be represented by AD in magnitude and direction.

Therefore, forces P, Q will be represented in magnitude and direction by AB, AD.

But by the parallelogram of forces, AC represents the resultant of forces AB and AD

∴ AC *represents in magnitude and direction the resultant of P and Q.*

Hence AC, CA represent in magnitude and direction the resultant of P, Q, R.

But forces represented by AC, CA balance each other (*i e.* give zero as their resultant) since the forces act on the same particle.

∴ The forces P, Q, R are in equilibrium.

The student should note carefully the fact that the sides of the triangle ABC are not *the lines of action of the forces* P, Q, R; *i. e.* while they represent the forces in *magnitude and direction*, they do not do so in *position.*

Corollary.—It is obvious that the force represented by AC is the resultant of forces represented in magnitude and direction by AB and BC. Hence the Triangle of Forces may also be enunciated as :—

If two forces acting on a particle be represented in magnitude and direction by the two sides of a triangle taken in the same order, then the resultant will be represented in magnitude and direction by the third side taken in the opposite order :—

In vector notation, this may be expressed by the equation $\overline{AB}+\overline{BC}=\overline{AC}$.

i. e. the resultant of AB, BC is represented by AC in magnitude and direction (not in line of action) *observe* that in Geometry, where the lines are *scalar quantities* $AB+BC>AC$.

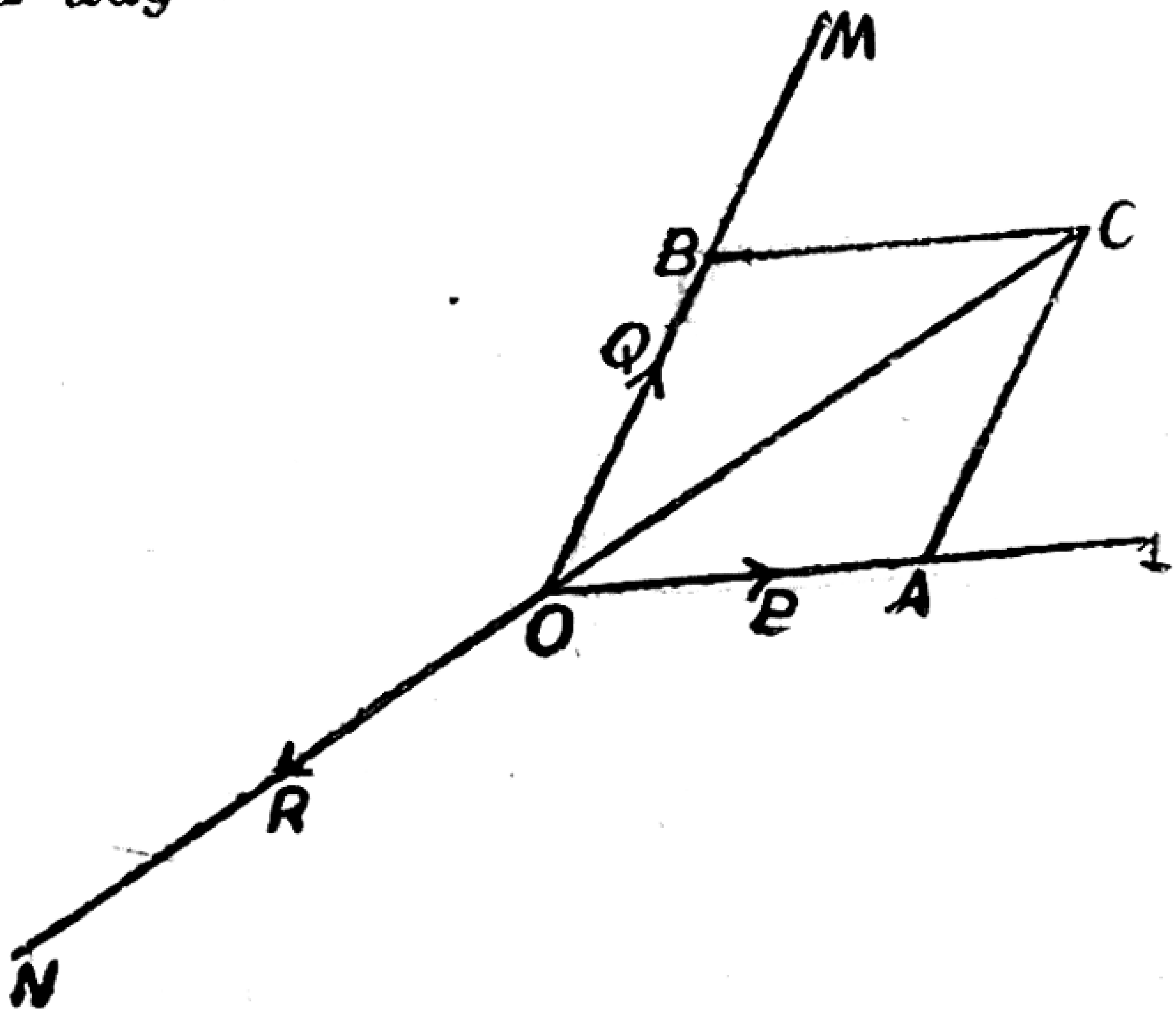
This is practically another way of enunciating the parallelogram of forces.

Note 1. If the two forces represented by AB and BC also *acted along these sides,* their resultant would *act at B along a line parallel to AC.*

Note 2. We may extend the corollary to any multiple of the forces and obtain the equation as follows :

$$l.\,\overline{AB}+l.\,\overline{BC}=l.\overline{AC}.$$

24. Converse of the Triangle of Forces.—*If three forces acting on a particale be in equilibrium, they can be represented in magnitude and direction by the sides of any triangle drawn in such a way as to have its sides parallel to the forces~~, taken~~ in order.*



Let the three forces P, Q, R acting on the particle at O along OL, OM, ON, be in *equilibrium.*

Cut off lengths OA, OB from OL, OM to represent forces P and Q respectively.

Complete the ||^m OACB, and join OC.

Now, by the parallelogram of forces,

OC represents the resultant of forces P and Q represented by OA and OB.

But P, Q, R are in equilibrium,

i. e. OC and R acting at O are in equilibrium.

∴ Force R acting along ON is equal and opposite to OC

∴ R *is represented by CO (i e.* OC *reversed).*

Also P ,, ,, ,, OA (*construction*)

and Q which is represented by OB may be represented by AC which is equal and ∥ to OB.

Thus the three forces P. Q. R acting at O are represented respectivley by OA, AC, CO, the three sides, taken in order, of the △ OAC.

Again, any triangle whose sides are parallel to the directions of the forces P, Q, R will have its sides parallel to the sides of the △ OAC and will, therefore, be equiangular and similar to it. Hence, the sides of *any triangle* drawn in the way described will represent the forces in magnitude and direction.

Note 1—This theorem can be extended and enunciated as :—*If three forces acting on a particle be in equilibrium, they can be represented in magnitude by the sides of a triangle drawn in such a way that its respective sides make a constant angle with the direction of the forces.*

This triangle is similar to the triangle, OAC, because each of its sides make a constant angle with the corresponding sides of OAC and, consequently, the angle between any two of its sides will be equal to the corresponding angle of the triangle OAC.

Or, If the new triangle be turned in the plane through an angle equal to the constant angle but in the opposite direction, the triangle so obtained will have its sides parallel to those of the triangle OAC.

Note—*As a special case, the constant angle may be taken to be a right angle.*

The theorem, in this special case, is called the "**Perpendicular Triangle of Forces**" and may be enunciated as follows :

"*If three forces acting at a point can be represented in magnitude by the sides of a triangle, and if the directions of the forces are perpendicular to the sides and the forces all act outwards or all act inwards, then they are in equilibrium.*"

Note 2.– When the magnitudes of three forces are given, a triangle can be constructed to represent the forces (except when one of the forces is greater than the sum of the other two). *The sides of the triangle give the relative directions of the forces*, in order that they may be in equilibrium. Thus, by this theorem, we have a method for finding (*i*) the *relative magnitudes* of three forces in equilibrium when their directions are given and (*ii*). the *relative directions* when magnitudes are given.

25. Lami's Theorem :—*If three forces acting on a particle keep it in equilibrium, each is proportional to the sine of the angle between the other two.* This theorem is the trigonometrical form of the converse of the "Triangle of Forces."

Let the three forces be P, Q, R, acting on O. Let P and Q be represented by OA and OB [*vide* figure Art. 24]. Then, since R is equal and opposite to the resultant of P and Q, on completing the parallelogram OACB, we see that R is represented by CO, while P and Q are represented by OA and AC, respectively.

Since the sides are proportional to the sides of the opposite angles, and forces are proportional to the sides, we have

$$\frac{P}{\sin OCA}=\frac{Q}{\sin COA}=\frac{R}{\sin OAC}.$$

Now $\sin OCA = \sin BOC = \sin MON$,

[supplementary angles]

$\sin COA=\sin LON$, and $\sin OAC=\sin LOM$.

$$\text{Hence } \frac{P}{\sin MON}=\frac{Q}{\sin LON}=\frac{R}{\sin LOM}.$$

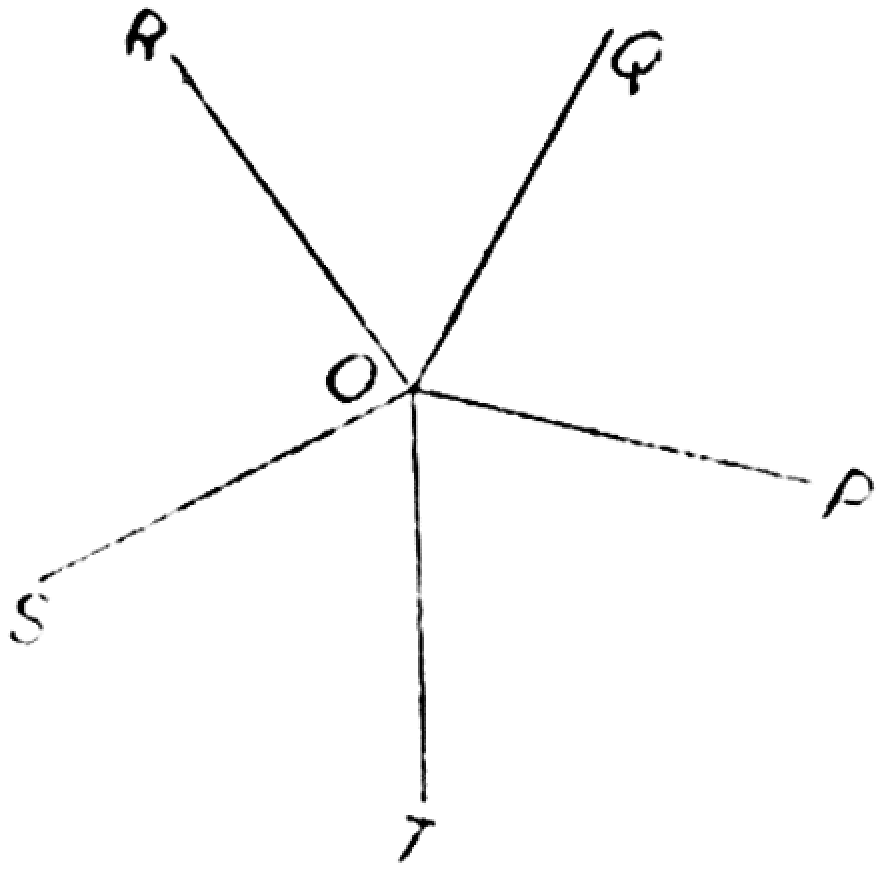
$$\textit{i.e.} \frac{P}{\sin (Q,R}=\frac{Q}{\sin (R,P)}=\frac{R}{\sin (P,Q)}.$$

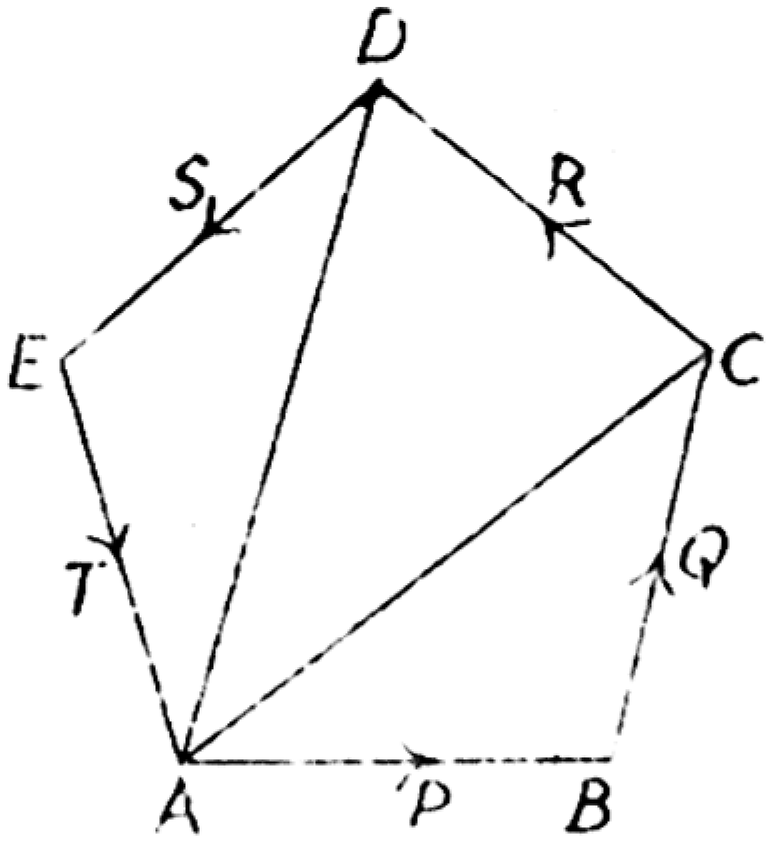
Where (Q.R), (R,P). (P,Q) denote the angles between the forces named by the letters.

The above theorem gives the form in which parallelogram of forces" was enunciated by Lami.

26. Polygon of Forces.—*If any number of forces,* **acting on a particle,** *be represented in magnitude and direction by the sides of a polygon,* **taken in order,** *the forces shall be in equilibrium.*

Let the forces P, Q, R, S, T acting on the particle O be represented in magnitude and direction by the sides AB, BC, CD, DE, EA, of the polygon ABCDE.



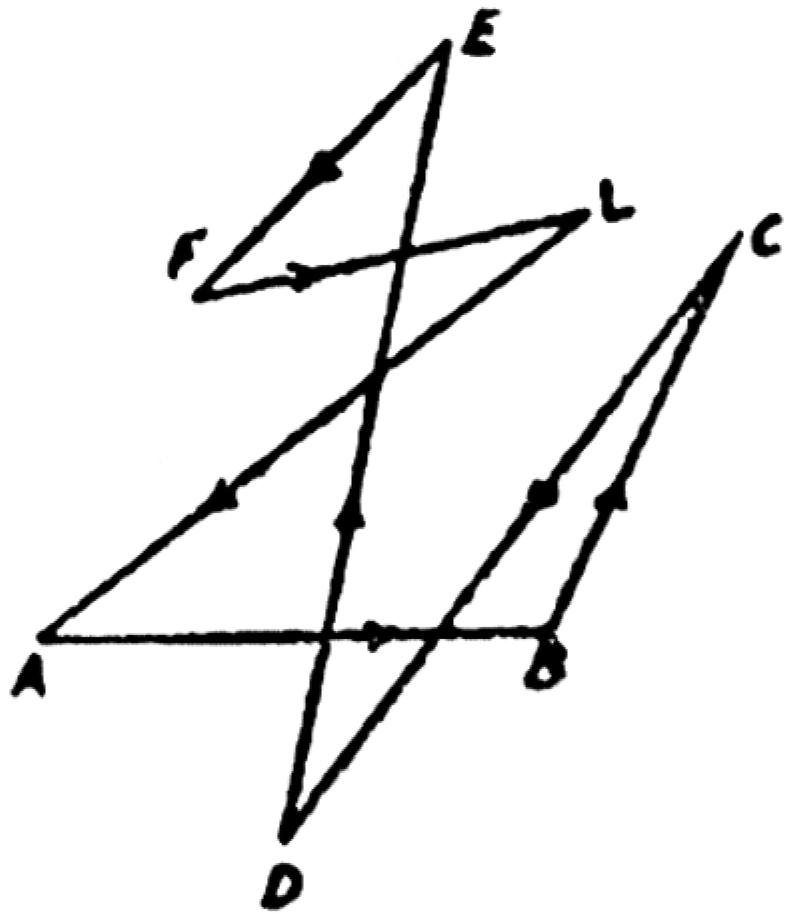


Draw AC, AD,

The resultant of P and Q in magnitude and direction
≡ resultant of AB, BC ≡ AC (*cor. to Triangle of Forces*)
∴ resultant of P, Q, R ≡ resultant of AC, CD ≡ AD.
∴ ,, P, Q, R, S ≡ ,, ,, AD, DE ≡ AE.
Hence ,, P,Q,R,S,T ≡ ,, ,, AE+EA ≡ O.

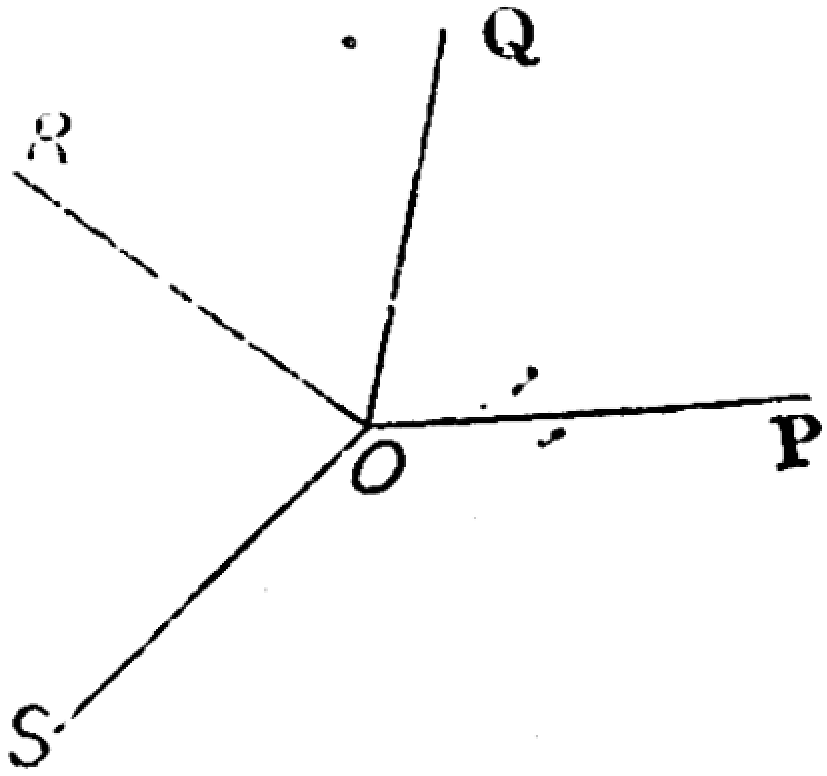
∴ the forces P, Q, R, S, T are in equilibrium. By adopting a similar method of proof, we can prove the theorem for any number of forces. Thus the general theorem of the polygon of forces is established.

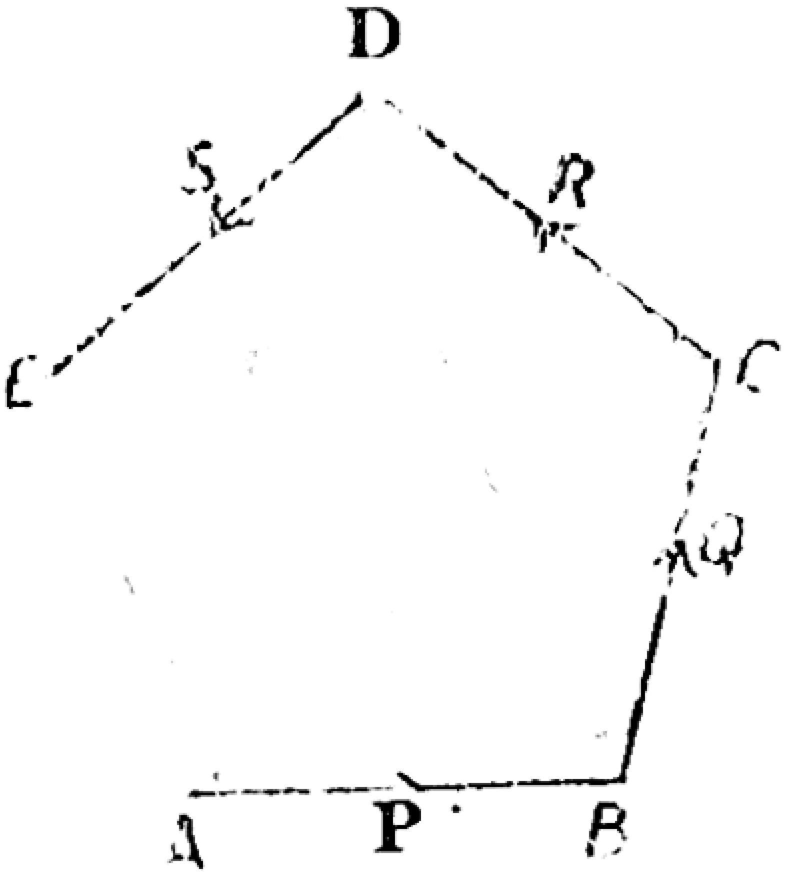
Note—It is easily seen from the method of proof employed above that the polygon may have its sides crossing each other or it may have its sides in different planes.



27. Graphical construction of the resultant of any number of forces acting on a particle.

The polygon of forces suggests the following method for finding the resultant of any number of forces P, Q, R, S etc., acting on a particle at O. From any point A draw AB parallel to the direction of P and also representing P in magnitude.





From the end point B, draw BC parallel and proportional to Q, similarly, draw CD, DE parallel and proportional to R and S.

(i) If the figure ABCDE thus constructed is an *unclosed polygon*, then the line AE required to close the polygon will represent in magnitude and direction (not in line of action) the resultant of P, Q. R, S acting at O and represented by AB, BC, CD, DE.

In vector notation, $\overline{AB}+\overline{BC}+\overline{CD}+\overline{DE}=\overline{AE}$.

(ii) If the figure ABCDE happens to be a *closed polygon* on account of the points A and E (the initial and the terminal points) coinciding with each other then, since AE=O, the resultant is zero and *the system of forces* P, Q, R, S *acting at O is in equilibrium*.

28. Converse of the Polygon of Forces.—We can enunciate the converse of the *polygon of forces* in the following form :

If any number of forces, acting on a particle, be in equilibrium, they can *be represented in magnitude and direction by a (closed) polygon, taken in order.*

This form of the Converse is true

Let the forces P, Q, R, S......acting on a *particle at O, be in equilibrium.*

From any point A, draw the figure ABCDE......as in Art, 27 so that, AB, BC, CD, DE......are parallel and proportional to P, Q, R, S......

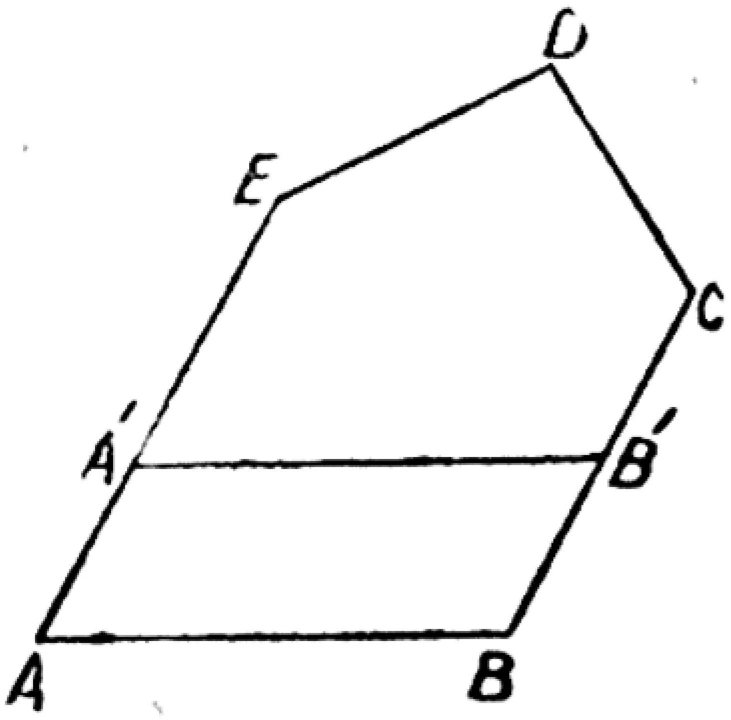
Then ABCDE......must be a closed polygon.

For, if ABCDE......be not a closed polygon, the forces P, Q, R, S......(as represented by AB, BC, CD, DE......) will have a resultant represented by the line drawn from A to close the polygon.

This is contrary to hypothesis.

29. *The converse of the Polygon of Forces is less general than that of the triangle of Forces.* In the case of converse of the Triangle of Forces, we have seen that the sides of *every triangle* which represent the forces in directions also represent the forces in magnitude, all equiangular triangles being similar.

But in the case of the converse of the Polygon of Forces, it is **not true** that "the sides of *every polygon*, taken in order, and drawn parallel to the concurrent forces in equilibrium will also represent the forces in magnitude"; because it does not follow that any two polygons which have their sides parallel, should also have their sides proportional in as much as equiangular polygons are not necessarily similar. This is evident from the adjoining figure where (A′B′ being drawn parallel to AB) the polygons ABCDE, A′B′CDE, though equiangular are not similar (*i. e.* their sides are not proportional),



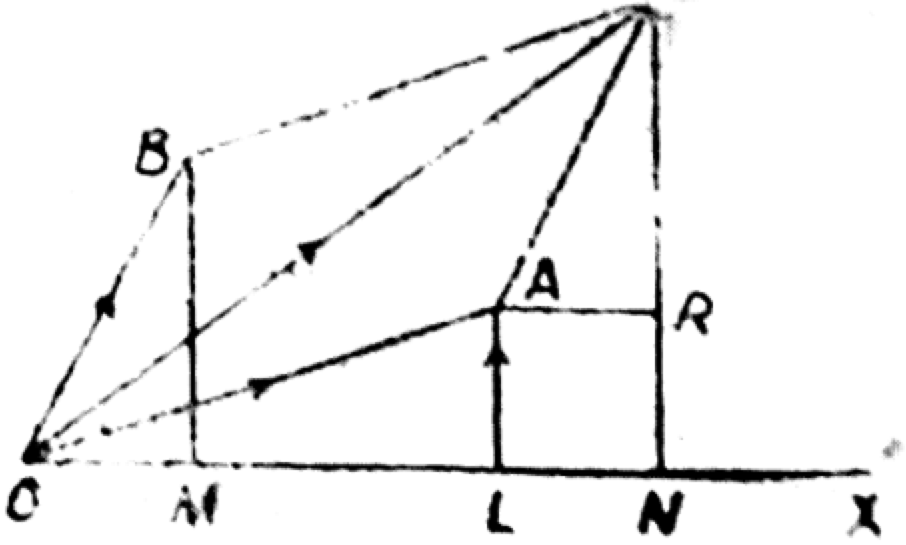
for obviously $\frac{BC}{CD} = \frac{B'C}{CD}$.

30. To prove that the algebraic sum of the resolved parts of two forces in a given direction is equal to the resolved part of the resultant in that direction.

Let the two forces be represented by OA and OB respectively, and the resultant by diagonal OC of the parallelogram OACB.

Let OX be the direction along which they are to be resolved.

From A, B, C, draw the perpendiculars AL, BM, CN upon OX, and from A again, perpendicular AR upon CN.



Then OL, OM, and ON are the resolved parts along OX of OA, OB, and OC, respectively. So *we have to prove that* OL+OM=ON.

Now the triangles OMB and ARC have their sides parallel and OB=AC.

They are congruent; Hence OM=AR=LN.

∴ OL+OM=OL+LN=ON, which proves the theorem

Remarks.—1. The equality OM=LN follows at once from the fact that they are the orthonogal projections on OX of two equal and parallel lines OB and AC. It is not necessary to prove the congruence of the △S OMB, ARC for this purpose.

2. If ∠BOX be obtuse M will fall on the left of O in XO produced and N will fall between O and L. It is easy to see by drawing a figure that OL−OM=ON. This shows that the *algebraic sum* (which also includes the arithmetical difference) of OL and OM is ON in this case also.

3. If P, Q, R be respectively the forces represented by OA, OB, OC, and α, β, θ be the angles made by them with OX, then we have the formula $P\cos\alpha + Q\cos\beta = R\cos\theta$.

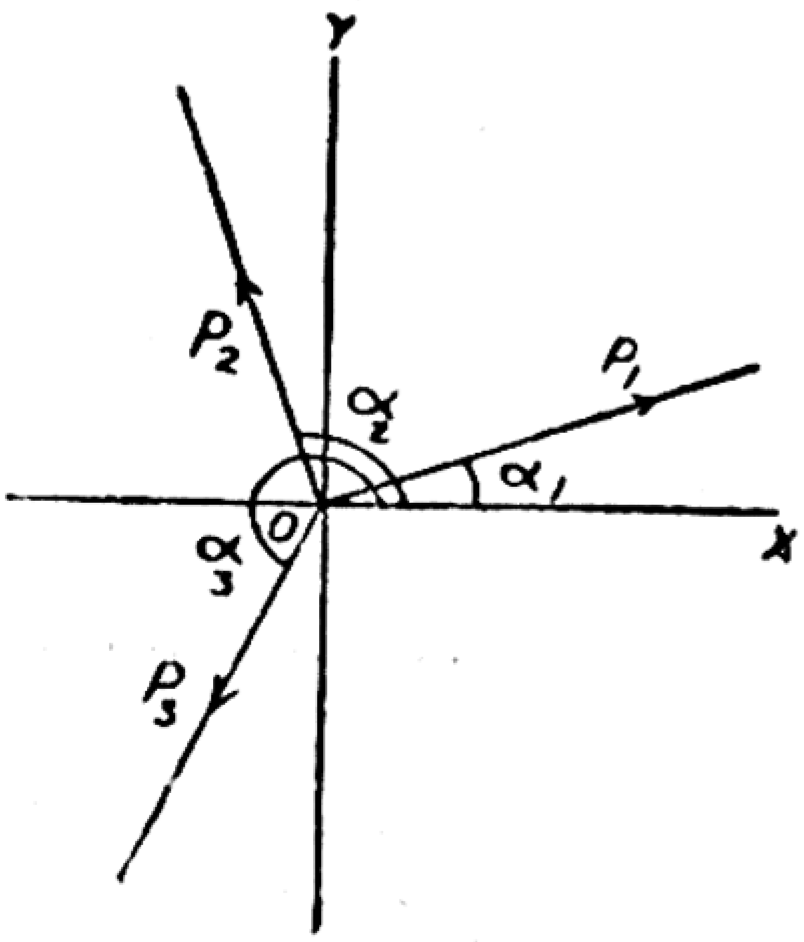
31. Generalised theorem for resolving any number of forces along a given direction. Let P, Q, R be any number of forces. Let F_1 be the resultant of P and Q, F_2 the resultant of F_1 and R, and so on.

Then the resolved part of P along OX
+ ” ” Q ” ” = resolved part of F_1 along OX.
Also ” ” F_1 ” ”
+ ” ” R ” ” = ” F_2 ” ”
Hence ” ” P ” ”
+ ” ” Q ” ”
+ ” ” R ” ” = resolved part of their resultant along OX.

Similarly, the theorem can be extended to apply to any number of forces.

32. Analytical conditions for equilibrium of any number of concurrent coplanar forces.

(*i*) *To find the resultant of concurrent forces.*



Before finding conditions of equilibrium we shall find *the resultant of such forces acting upon a particle.*

Let the forces P_1, P_2, P_3 .. make angles α_1, α_2, α_3...with any line **OX**, taken as x-axis.

Resolving the forces along **OX**, and the perpendicular line OY, we have (by Art. 31.)

$$X = P_1 \cos\alpha_1 + P_2 \cos \alpha_2 + P_3 \cos\alpha_3 + \ldots\ldots = \Sigma P \cos\alpha.$$

$$Y = P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + P_3 \sin \alpha_3 + \ldots = \Sigma P \sin \alpha,$$

where X, Y are resolved parts of the resultant.

If the resultant F be supposed to make an angle θ with OX, $F \cos \theta = X$, or $\Sigma P \cos \alpha$
and $F \sin \theta = Y$, or $\Sigma P \sin \alpha$.

$$\text{Thus } F = \sqrt{X^2 + Y^3} \text{ and } \tan \theta = \frac{Y}{X} = \frac{\Sigma P \sin\alpha}{\Sigma P \cos\alpha} \ldots (1).$$

So we have *the magnitude and direction of the resultant.*

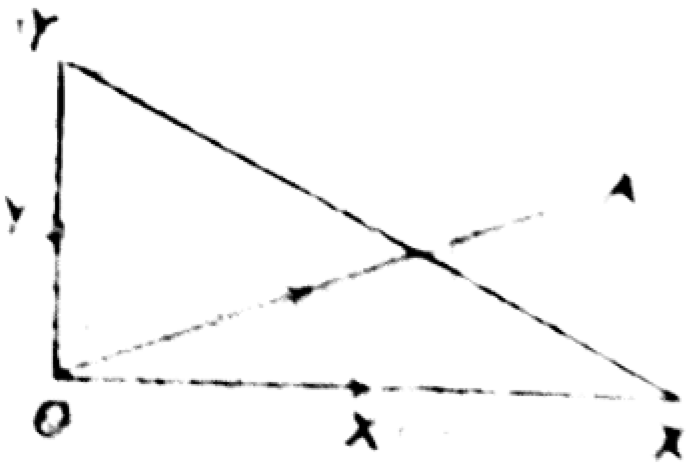
(*ii*) *To find the conditions of equilibrium.*

Since the magnitude of the resultant is $\sqrt{X^2 +}$ Y^2 resultant vanishes, only if X and Y separately vanish. Hence the *analytical conditions for the equilibrium of any number of concurrent co-planar forces are* :—

$$\Sigma P \cos \alpha = 0 \text{ and } \Sigma P \sin \alpha = 0 \qquad \ldots(2)$$

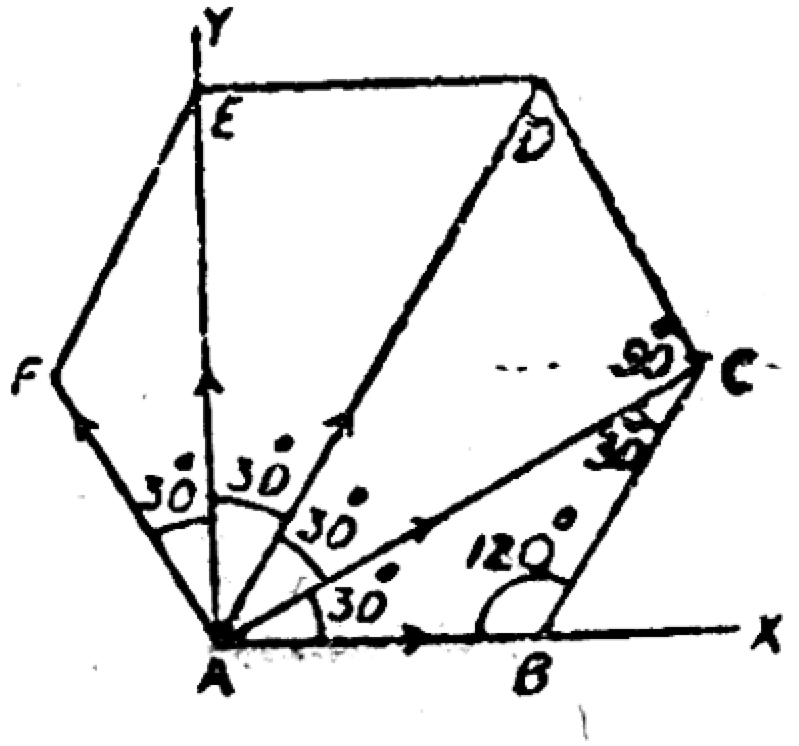
i.e. the algebraic sum of the resolved parts of the forces along each of two perpendicular lines vanishes separately.

Observation. It may be pointed out here that resolved parts of the forces are taken along two *perpendicular* lines *only as a matter of convenience.* We could as well take them along OX and any other line OA. For, since the sum of the resolved parts along OX vanishes, then either the resultant F=0 or F acts perpendicularly to OX; similarly since the sum of the resolved parts vanishes along OA, either F=0 or F acts perpendicularly to OA. Now F could not act in two different directions;



$\therefore$ F=0 and equilibrium exists.

Ex.—Five forces are represented in a regular hexagon ABCDEF by AB, AC, AD, AE, and AF, find in any manner, the resultant force.



First Method. The resultant of AB and AF is 2AB cos 60° along AD, *i.e.* AB along AD.

The resultant of AC and AE= 2AC cos 30°=AC$\sqrt{3}$, or 3 AB. along AD.

Also AD=2AB $\therefore$ The final resultant is 6 AB along AD.

Second Method. For obtaining the resultant *analytically*, we should first choose suitable axes. Here AE is perpendicular to AB.

Hence AB and AE may be chosen as axes of x and y respectively.

Resolving along AX,

$$X = AB + AC\cos 30° + AD\cos 60° + AF\cos 120°$$

(AE has no resolved part along AB)

$$= AB + \sqrt{3}AB\frac{\sqrt{3}}{2} + 2AB.\tfrac{1}{2} - AB.\tfrac{1}{2} = 3AB.$$

Resolving along AC,

$$Y = AC\sin 30° + AD\sin 60° + AE + AF\sin 120°.$$

$$= AB\frac{\sqrt{3}}{2} + 2AB\frac{\sqrt{3}}{2} + AB\sqrt{3} + AB\frac{\sqrt{3}}{2} = AB.\,3.\sqrt{3}.$$

$$\therefore F^2 = \{\,3^2 + (3\sqrt{3})^2\,\}\,AB^2 = 36AB^2, \text{ or } F = 6AB.$$

$$\text{Also } \tan\theta = \frac{Y}{X} = \sqrt{3}\ ,\therefore\ \theta = 60°.$$

The student should note that AF sin 120°=AF cos 30° and it is simpler to put it directly by multiplying AF by cosine of the angle between AF and AE.

Exercises II

1. Three forces acting on a particle keep it in equilibrium; if the angles between the first and the second force be 150°, and between the second and third be 120°, show that

they can be represented by the sides of a right angled triangle. If the third force be P, show that the first and second are $P\sqrt{8}$ and 2P respectively.

2. If three forces which acting on a particle keep it in equilibrium, be inclined successively to each other at 120°, show that they can be represented by the sides of an equilateral triangle, and must be equal.

Conversely, if the forces be given equal, show that they must be equally inclined.

✓ 3. Three forces of 12 lbs., 35 lb. and 37 lbs. acting on a particle keep it in equilibrium; show that the first two forces are at right angles to each other .

4. Find the relative directions of three forces which acting on a particle keep it in equilibrium, if their relative magnitudes are 8 : 15 : 17.

5. Give a geometrical construction to resolve a given force into two given components. When does the construction fail ?

✓ 6. Find by geometrical construction if (*i*) the forces of 5, 7, 12 lbs. (*ii*) the forces of 15 17, 36 lbs., and (*iii*) forces of 5, $5\sqrt{3}$ 10 lbs., acting on a particle will keep it in equilibrium. If they do so, find their directions.

7. A force of $25\sqrt{2}$ lbs., and two other forces acting on a particle are in equilibrium. If the angle between the first and second be 75° and between the second and third 135°, determine the other forces.

8. Four forces acting at the centre of a square are represented by its sides, taken in order. Show that they are in equilibrium.

9. Forces of 5, 6, 8, and 9 lbs. act from the centre O of a square ABCD, along OA, OB, OC, OD, respectively, and forces of $5\sqrt{2}$ lbs., $6\sqrt{2}$ lbs., $8\sqrt{2}$ lbs., and $9\sqrt{2}$ lbs., act at O parallel to AB, BC, CD, DA, respectively; show, in any manner, that the resultant of all the forces is $3\sqrt{2}$ lbs. parallel to BA.

10. Five equal forces act at the centre of a regular hexagon parallel to five sides, taken in order; find the resultant completely.

11. If ABCD is a rectangle such that the diagonal AC is 2AB and forces of 2 lbs., 4 lbs., and 8 lbs., act at A along AB, AC, and parallel to BD, respectively; find the resultant.

12. In a regular hexagon ABCDEF, five forces acting at A, are represented by 2AB, 3AC, 4AD, 3 AE and 2 AF; find the direction and magnitude of the resultant.

13. Forces of 1, 2, 3, 4, 5, 6 lbs. act at the centre of a regular hexagon parallel to the sides, taken in order; find the resultant.

14. Show that the resultant of forces 7, 1, 1, and 3 lbs. acting from an angular point of a regular pentagon towards the remaining angular points, taken in order, is $\sqrt{71}$.

[*Hint*:—Draw lines from the angular point (at which the forces are acting) parallel to the opposite side and perpendicular to it, and take these as axes. *Otherwise;* First compound the forces 1, 1; 3, 3, and then the resultant with the remaining force 4.

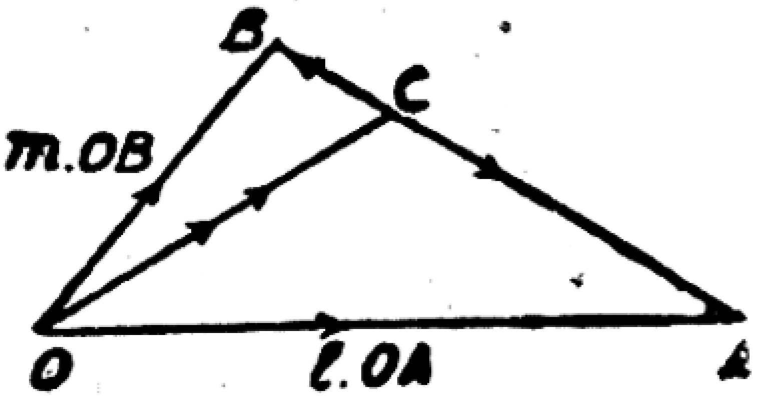
15. Show that the resultant F of three concurrent forces P, Q, R in the same plane is given by the formula $F^2=P^2+Q^2+R^2+2PQ\cos(P, Q)+2QR\cos(Q, R)+2RP\cos(R,P)$.
where cos (P, Q) stands for cosine of the angle between the forces P and Q.

[*Hint*:—Resolving along and perpendicular to the direction of P, we get $F\cos\theta=X=P+Q\cos(P, Q)+R\cos(R,P)$; $F\sin\theta=Y=0+Q\sin(P, Q)+R\sin(R, P)$, square and add.

16. ABCD is a square, each side being 17·8 inches and E is the midpoint of AB. Forces 7, 8, 12, 5, 9 and 6 lbs. act on a body in the lines and directions AB, EC, BC, BD, CA and DE respectively. Find the magnitude and position of the single force required to keep the body in equilibrium.

[P.U. 1944]

33. Theorem. *If two forces acting on a particle are represented by l OA and m.OB, then their resultant will be represented by (l+m) OC, where C is a point in AB such that l.AC=m.CB*



From the triangle OCA, the force represented by *l*.OA is equivalent to forces represented in magnitude and direction, by *l*.OC and *l*.CA. Similarly the force *m*.OB is equivalent to forces *m*.OC and *m*.CB (in magnitude and direction.)

Since $l.CA=m.CB$, the forces represented by $l.CA$ and $m.CB$, being in opposite directions, neutralize each other, and we are left with forces $l.OC$ and $m.OC$ which give the resultant

$$(l+m).OC.$$

Note 1.—The forces $l.CA$ and $m.CB$ are acting actually at O.

Note. 2.—This theorem is of considerable importance. If l is large in comparison with m it, will not be possible to draw lengths to represent the forces, on the same scale in the usual manner, and hence the resultant cannot be represented by the diagonal of a parallelogram; the present theorem overcomes the difficulty easily.

Note 3.—In vector notation, the above result may be expressed by the following equation.

$$l.OA+m.\overline{OB}=(l+m)\,\overline{OC}.$$

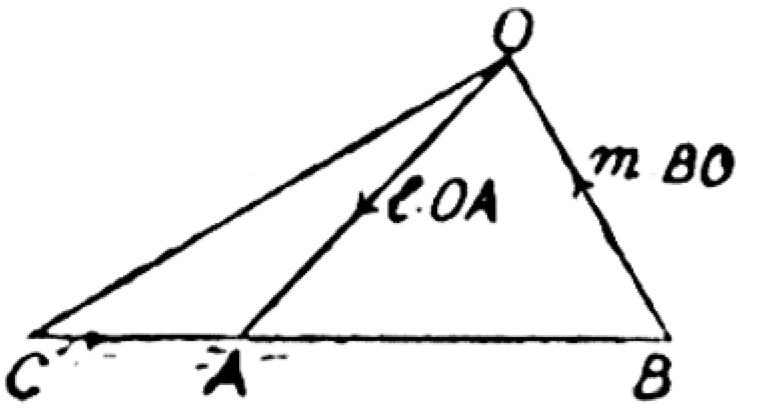
By changing the directions of the forces, we easily deduce

$$l.AO+m.BO=(l+m)\,CO$$

If $l=m$, C becomes the middle point of AB and the forces $l.OA$ and $l.OB$ have the resultant $2l.OC$. Since 2OC will be the diagonal of the parallelogram constructed with OA and OB as adjacent sides, the theorem reduceds to the parallelogram of forces.

If $l=m=1$, the resultant of OA and OB $=$ 2OC where C is the midpoint of AB.

34 If the forces be $l.OA$ and $m.BO$ (instead of $m.OB$) and $l>m$, the resultant is $(l-m)\,OC$, where C divides $\overrightarrow{BA}$ *externally* such that $l.AC=m.BC$ $\left(i.\ e.\ \dfrac{BC}{AC}=\dfrac{l}{m}\right)$.



We observe that the forces are acting at O, and while one of them diverges from O, the other converges to O.

We have

$l.OA \equiv l.OC$ along OC and $l.CA$ acting at O parallel to AB.

Similarly, $m.BO \equiv m.BC$, acting at O parallel to AB and $m.CO$ along CO or $-m.OC$

$$\therefore \left.\begin{matrix} l.OA \\ \text{and } m.BO \end{matrix}\right\} \equiv \begin{cases} (i)\ (l-m)\ OC \text{ along } OC, \\ (ii)\ l.CA \text{ and } m.BC \text{ acting at } O \\ \text{parallel to } AB. \end{cases}$$

But the forces *(ii)* namely $l.CA$ and $m.BC$ neutralize each other, since $l.AC=m.BC$ (given)

Hence we are left with the force $(l-m)$ OC, which is the resultant.

Note. If $l<m$, the resultant is $(m-l)$ OC where C divides AB *externally* in the ratio $\frac{l}{m}$ so that $\frac{BC}{AC}=\frac{l}{m}$. The point C in this case is on the right of B in AB produced. **Observe** that the external point of division C is nearer to A or B according as $l>$, or $<m$.

In the vector notation, the proof given in 34 may be simplified as follows :

$$l.\overline{OA} = l.\overline{OC}+l.\overline{CA}$$

$$m.\overline{BO} = m.\overline{BC}+m.\overline{CO}$$

$\therefore$ $l.\overline{OA}+m.\overline{BO} = l.\overline{OC}+m.\overline{CO}+l.\overline{CA}+m.\overline{BC} = l.\overline{OC}+m.\overline{CO}$, (since $l.\overline{CA}$ and $m.\overline{BC}$ neutralise each other, as they are given equal and act in opposite sense)
$= l.\overline{OC}-m.\overline{OC}$ ($\overline{CO}=-\overline{OC}$) $=(l-m)\,\overline{OC}$.

Remarks

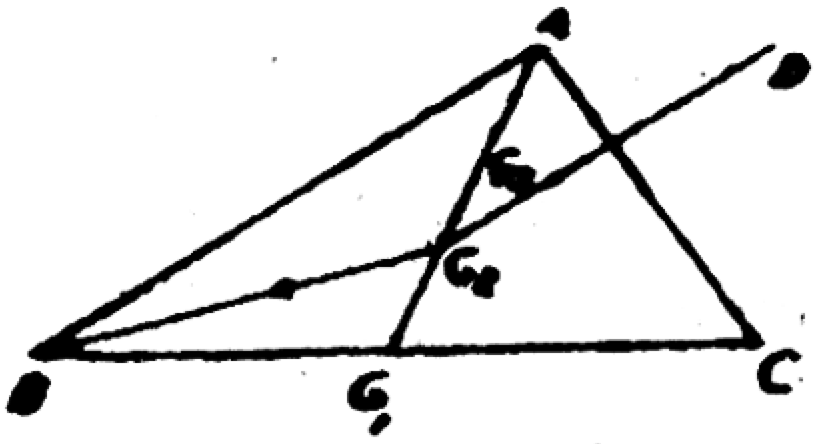
If $l=m$, we have

(*i*) $l.\overline{OA}+m.\overline{BO}=l.\overline{OA}+l.\overline{BO}=l\,(\overline{BO}+\overline{OA})=l.\overline{BA}$ which is merely a re-affirmation of the theorem of the Triangle of Forces.

(*ii*) also $l.\overline{AC}=m.\overline{BC}$. gives CA=BC.

Hence C must be at an infinite distance, *i. e.* OC is parallel to BA.

35. Centroid.—In goemetry, the point of intersection of the medians of a triangle has been defined as the *centroid* of the triangle. In any triangle ABC, if G_1 be the middle point of BC, the centroid G_2 is such that $2G_1G_2=G_2A$.



Generally, G_1 is defined as the centroid of points B and C, and G_2 as the centroid of points A, B, C. If a fourth point D is joined to G_2 and DG_2 divided at G_3, such that $3G_2G_3=G_3D$, G_3 will be the centroid of points A, B, C, D ; and so on.

10. If forces P, Q, R, S, acting along the sides AB, CB, CD, AD, respectively, of a quadrilateral ABCD be in equilibrium, show that

$$\frac{P}{AB} : \frac{Q}{CB} :: \frac{S}{AD} : \frac{R}{CD}. \qquad \text{(P. U. 1935).}$$

[*Hint* : Representing P by λ.AB, Q by μ. CB, R by λ'. CD, and S by μ'. AD, prove that $\lambda\lambda' = \mu\mu'$.]

11. A line cuts the lines of action of three concurrent forces P, Q, R in A, B, C, respectively. If R be the resultant of P and Q, show that

$$\frac{P}{OA} + \frac{Q}{OB} = \frac{R}{OC}. \qquad \text{(P. U. 1928.)}$$

Sol. Resolve the forces perpendicular to ABC.

Or, representing P by λ.OA, Q by μ.OB, R will be represented by $(\lambda+\mu)$.OC. Hence

$$\frac{P}{OA} + \frac{Q}{OB} = \lambda + \mu = \frac{R}{OC}.$$

This is then only another way of stating the theorem of Art. 33.

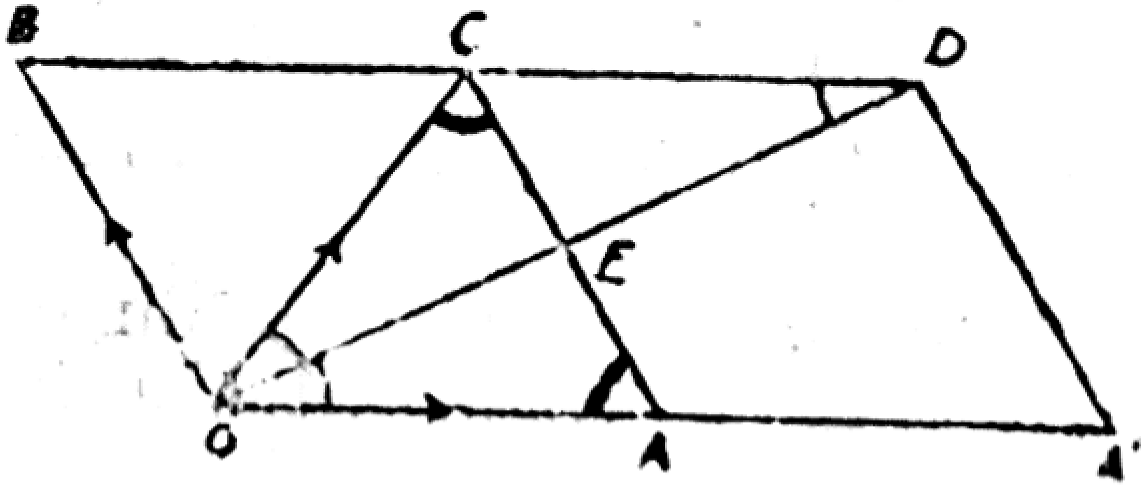
12. ABCD is a quadrilateral ; L_1, L_2 are the middle points of AB, CD ; M_1, M_3 of AD, BC ; and N_1, N_2 of AC, BD. Find in different ways, the resultant of forces represented by PA, PB, PC, PD, where P is any point ; hence show that L_1, L_3 ; M_1, M_3 ; N_1, N_2 are concurrent and have the same point of bisection.

MISCELLANEOUS WORKED OUT EXAMPLES

36 **Ex. 1.** Two forces act at such an angle that the resultant is equal to one of the forces, show that if that force be doubled the resultant will become perpendicular to the other force.

First Method.

Let the forces be represented by OA and OB and have the resultant OC, the diagonal of the parallelogram, so that OC=OA.



When the force along OA is doubled, let it be represented by OA' such that OA'=2 OA.

Completing the parallelogram OA′ DB, the resultant is represented by OD.

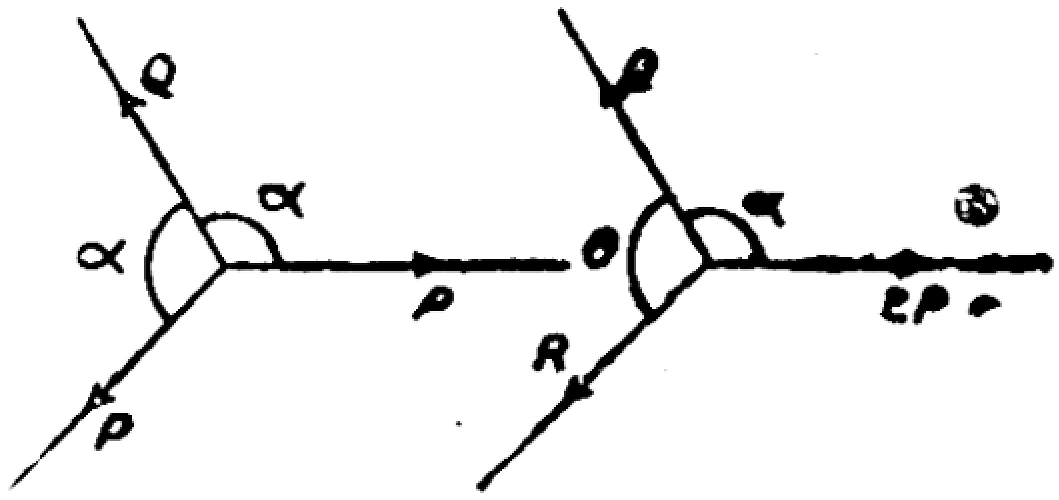
Also OC=OA=AA′ =CD.

∴ ∠COD=∠CDO=∠DOA, so that OE bisects the vertical angle of the isosceles triangle OAC ∴ OE is perpendicular to AC or to OB, the direction of the other force

This method is geometrical. It will be instructive to see how the problem could also be solved by Lami's Theorem.

Second Method.

In case (*i*) the forces P, Q have the resultant P, so that P, Q and P (opposite to the direction of the resultant) are in equilibrium.



(*i*) (*ii*)

$$\therefore \quad \frac{P}{\sin\alpha}=\frac{P}{\sin\alpha}=\frac{Q}{-\sin 2\alpha},$$

$$\text{whence} \quad \frac{P}{Q}=-\frac{1}{2\cos\alpha}. \qquad \text{......(A)}$$

In case (*ii*), let 2P and Q be balanced by R making an angle θ with Q.

$$\therefore \quad \frac{R}{\sin\alpha}=\frac{2P}{\sin\theta}=\frac{Q}{-\sin(\alpha+\theta)},$$

$$\text{whence} \quad \frac{P}{Q}=\frac{-\sin\theta}{2\sin(\alpha+\theta)}. \qquad \text{......(B)}$$

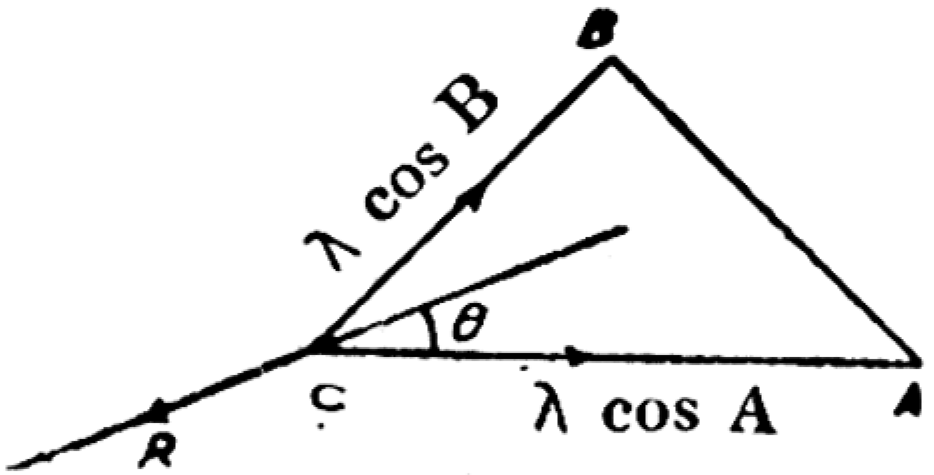
From (A) and (B), $\cos\alpha\sin\theta=\sin(\alpha+\theta)$,

or $\sin\alpha\cos\theta=0.$

Since $\sin\alpha\neq 0$, $\cos\theta=0$ or, $\theta=\frac{\pi}{2}$.

The special utility of the second method will be seen by applying it to the succeeding example taken from *Greaves' Statics.*

Ex. 2. Two forces act along the sides CA, CB of a triangle ABC, their magnitudes being proportional to cos A cos B. Prove that their resultant is proportional to sin C and that its direction divides the angle C into two parts $\frac{1}{2}(C+B-A)$, $\frac{1}{2}(C+A-B)$. (P. U. 1924)



If the resultant of the forces λ cos A, λ cos B be R, then R reversed is in, equilibrium with λ cos A and λ cos B.

$$\therefore \frac{R}{\sin C} = \frac{\lambda \cos A}{\sin (C-\theta)} = \frac{\lambda \cos B}{\sin \theta}.$$

whence $$\frac{\sin (C-\theta)}{\sin \theta} = \frac{\cos A}{\cos B}.$$

Now, as in the last example, we solve this equation for θ. Thus $\sin C \cot \theta - \cos C = \dfrac{\cos A}{\cos B}$,

or $$\sin C \cot \theta = \frac{\cos A}{\cos B} + \cos C$$

$$= \frac{\sin B \sin C}{\cos B} = \tan B \sin C.$$

Hence $\cot \theta = \tan B = \cot\left(\dfrac{\pi}{2} - B\right)$,

or $\theta = \dfrac{\pi}{2} - B = \frac{1}{2}(A+B+C) - B = \frac{1}{2}(A+C-B)$.

Hence $C-\theta = \frac{1}{2}(B+C-A)$.

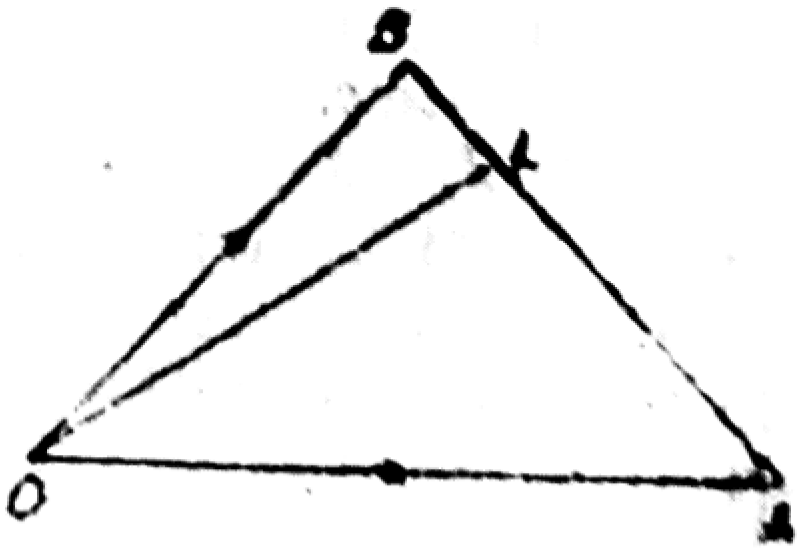
Also $$R = \frac{\lambda \cos B \sin C}{\sin \theta} = \lambda \sin C.$$

Ex. 3. Two forces acting along the sides OA, OB of a triangle OAB are represented by OA tan A and OB tan B; show that their resultant is represented by AB tan A tan B and act along the perpendicular from O on AB. (P.U. 1929).

Draw OL perpendicular to AB.

Since $\dfrac{AL}{BL} = \dfrac{OL/BL}{OL/AL} = \dfrac{\tan B}{\tan A}$,

L divides AB in the ratio tan B : tan A.



Hence the resultant of OA tan A and OB tan B acts along OL and is represented by

(tan A+tan B). OL [*Art.* 33.]

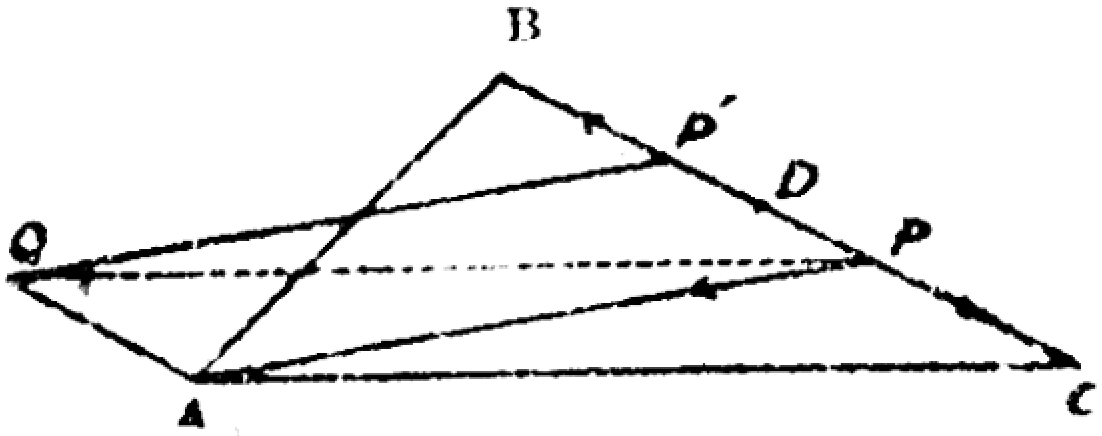
$$=\left(\frac{OL}{AL}+\frac{OL}{BL}\right).\ OL=\frac{AB.OL^2}{AL.BL}=AB\tan A\tan B.$$

Ex. 4. ABC is a triangle, and P is *any* point in BC. If PQ represent the resultant of the forces represented by PA, PB, PC, prove that the locus of Q is a straight line, parallel to BC. (P. U. 1923).

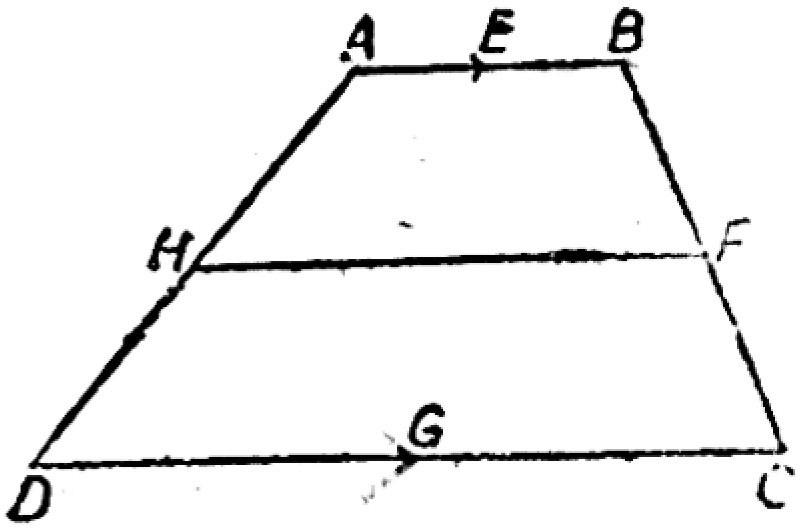
Let D be the middle point of BC.

Since, PB+PC=2PD, the resultant of PB and PC will be PP′, where DP′=PD.

But the resultant of forces represented by PA and PP′ is represented by the diagonal PQ of the parallelogram whose adjacent sides are PA and PP′. Thus Q lies on the straight line through A parallel to CB.



Ex. 5. The sides AB, BC, CD and DA of a quadrilateral ABCD are bisected at E, F, G, H, respectively. Show that if two forces parallel and equal to AB and DC act on a particle, then the resultant is parallel to HF and equal to 2HF.



Employing vector notation, we have

$$\overline{AB}=\overline{AH}+\overline{HF}+\overline{FB}$$

$$\overline{DC}=\overline{DH}+\overline{HF}+\overline{FC}$$

$$\therefore\ \overline{AB}+\overline{DC}=(\overline{AH}+\overline{DH})+2\overline{HF}+(\overline{FB}+\overline{FC})$$

$$=0+2\overline{HF}+0=2\overline{HF}$$

($\because\ \overline{AH}$ is equal and opposite to $\overline{DH}$ and so are $\overline{FB}$ and $\overline{FC}$.)

Exercises IV

1. If one force be double the other and they have a resultant equal to $\sqrt{3}$ times the smaller force, find the angle at which the component forces are inclined and also the angles which the resultant makes with them.

2. Two forces have the maximum resultant 23, but if they acted at right angles, the resultant would be 17 ; find the forces.

3. Forces P and Q act at such an angle that the resultant is doubled either when Q is doubled, or when the direction of Q is reversed. Find the ratio of the forces.

4. Two forces $P+Q$ and $P-Q$ make an angle 2α with one another, and their resultant makes an angle θ with the bisector of the angle between them. Show that
$P \tan \theta = Q \tan \alpha$. (*P. U. 1931.*)

5. Two forces P and Q have a resultant R, which makes a certain angle with P ; if P be increased by R while Q remains unchanged, show that the resultant will now make half the angle with the force $P+R$.

6. Two forces in the ratio $\sqrt{3} : \sqrt{2}$ act at angle 105° ; find by the application of Lami's Theorem, the angles which the resultant makes with the component forces and the proportional magnitude of the resultant.

7. The resultant of two forces P, Q, acting at an angle θ, is equal to $(2m+1)\sqrt{P^2+Q^2}$; when they act at an angle $\frac{\pi}{2}-\theta$, the resultant is equal to $(2m-1)\sqrt{P^2+Q^2}$; show that

$$\tan \theta = \frac{m-1}{m+1}.$$ (Greaves.)

8. If any three points A, B, C be taken in the circumference of a circle and forces P, Q, acting along AB and BC, respectively, have a resultant R acting along the tangent at B, show that $P : Q : R : : BC : AB : AC$.

9. The resultant of forces P and Q is R ; if Q be doubled, R is doubled ; whilst if Q be reversed, R is again doubled. Show that $P : Q : R : : \sqrt{2} : \sqrt{3} : \sqrt{2}$.

10. Prove that if O be the circumcentre of a triangle ABC, the resultant of the forces represented by OA, OB, OC is represented by OP, where P is the orthocentre.

Deduce the condition in order that forces represented by the straight lines drawn from the circumcentre of a triangle to the vertices, may balance.

11. Forces each equal to F act at a point parallel to the sides of a triangle ABC. If R be their resultant, prove that

$$R^2=F^2(3-2\cos A-2\cos B-2\cos C).$$

(*P. U. 1935*).

[*Hint*: Resolve the forces parallel and perpendicular to AB]

12. The resultants of forces P,Q acting at a certain angle is F, and that of P, R acting at the same angle is also F. The resultant of Q, R again acting at the same angle is G. Prove that if $P+Q+R=0$, $G=F$. (*P. U. 1938.*)

Sol. $F^2=P^2+Q^2+2PQ\cos\alpha$……(*i*)
$F^2=P^2+R^2+2PR\cos\alpha$……(*ii*)
$G^2=Q^3+R^2+2QR\cos\alpha$……(*iii*)

From (*ii*) and (*iii*), $F^2-G^2=(P-Q)\ (P+Q+2R\ \cos\alpha)$ (*iv*)
From (*i*) and (*ii*), $Q+R=-2P\ \cos\ \alpha=-P$
$\therefore\ \cos\ \alpha=\frac{1}{2}$ (or $\alpha=60°$).

Now substitute for cos α from this equation in (*iv*).

18. ABC is a triangle and D, E, F are the middle points of the sides; forces represented by AD, $\frac{2}{3}$ BE and $\frac{1}{3}$ CF act on a particle at the point where AD and BE meet. Show that the resultant is represented in magnitude and direction by $\frac{1}{2}$ AC, and that its line of action divides BC in the ratio 2 : 1. (*P. U. 1933*).

[*Hint*: Treat the force AD in two parts, using the Triangle of Forces.]

14. ABCD is a parallelogram and P any point in its plane. Through P, KL and MN are drawn parallel to the sides to meet the other sides in K, L : and M, N, respectively. Show that the resultant of the forces represented by PK, PL, PM, PN will be represented by 2 PO, where O is the centre of the parallelogram.

15. ABCD, A′B′C′D′ are two parallelograms, prove that forces acting at a point, proportional to, and in the same direction as AA′, B′B, CC′, D′D, will be in equilibrium. (Greaves).

16. From a point within a circle whose centre is fixed, straight lines PA_1, PA_2, PA_3, PA_4 are drawn to meet the circumference, all being equally inclined to the radius through P. Show that if these lines represent forces radiating from P, their resultant is independent of magnitude of the radius of the circle (Greaves ; *P. U. 1934.*)

37. We have seen that forces may appear in the form of *tension*, *attraction*, or *reaction*, (Art. 8). Now when a force appears as tension, and the string is tight, and the force acts directly or by passing over a smooth pulley, the tension is the same at all points of the string. If the force of tension acts through the agency of a rod, the rod should be *light* and *rigid*, in order that the tension may be the same all through.

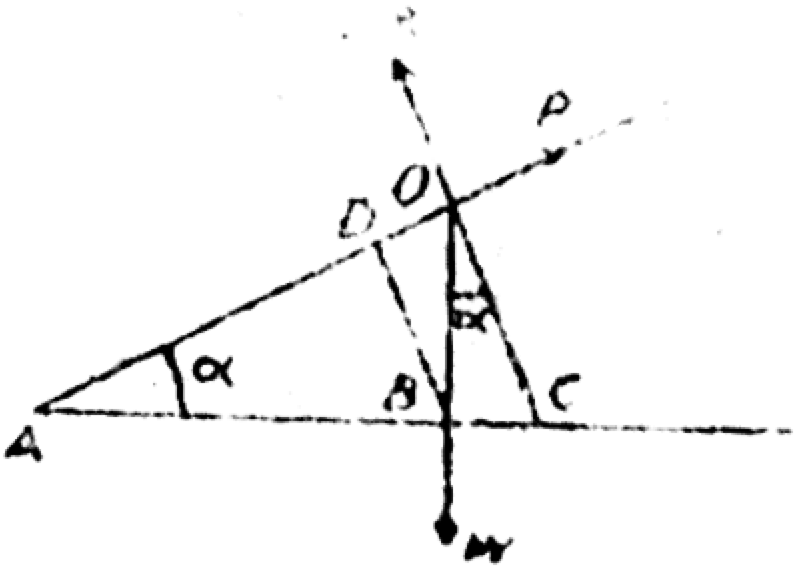
When a light string passes through a smooth ring, the tension will be the same in two parts of the string. But when a string is *knotted* to any weight, the two parts of the string will act as two different strings and the tension will not necessarily be the same. While, in the first case, the weight is free to slide on the string, in the second it is tied and has no freedom of movement.

There is always a limit to the strain which a string can bear ; when the tension exceeds this limit, the string breaks.

Again, when the force of reaction appears as the result of one body pressing against another, it is necessary that the bodies shall be rigid. Thus *rigidity* is an essential property of the bodies, with reference to which we consider motion or equilibrium. (See **Art. 6**).

38. We shall now illustrate the practical applications of the elementary principles of Statics discussed hitherto.

Ex. 1. Inclined Plane. *A particle of weight W rests on a smooth plane inclined to the horizon at an angle* α, *and is supported by a force acting along the line of greatest slope ; find the force and the reaction of the plane.*



Since the plane is smooth, the reaction of the plane on the particle is along the normal to the plane, so that if R be the reaction, and P the supporting force, R is perpendicular to P. O being the particle, W acts along the vertical OB, and if OC be the prolongation of R, it is obvious that

$\angle COB = \angle OAC = \alpha$.

Now P, R, W, acting at O, are in equilibrium,

$\therefore$ by Lami's Theorem, we have

$$\frac{R}{\sin\left(\frac{\pi}{2}+\alpha\right)} = \frac{P}{\sin(\pi-\alpha)} = \frac{W}{\sin\frac{\pi}{2}}$$

or $$\frac{R}{\cos\alpha} = \frac{P}{\sin\alpha} = W.$$

Hence $R = W\cos\alpha$ and $P = W\sin\alpha$.

Otherwise thus :—

P is acting along the plane and R perpendicular to it, hence the conditions of equilibrium are found easily by resolving W along these two directions,

whence $P = W\sin\alpha$ and $R = W\cos\alpha$.

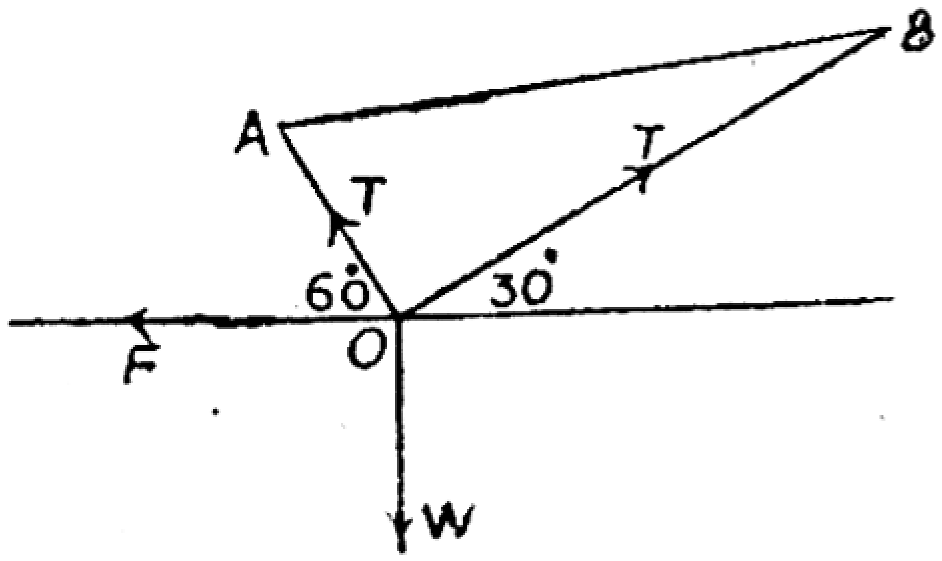
Or.—By drawing BD perpendicular to OA, we see that the forces can be represented by the sides of the triangle OBD.

$$\therefore \frac{P}{OD} = \frac{R}{BD} = \frac{W}{OB}, \quad i.\ e.\ P = \frac{OD}{OB}.\ W = W\sin\alpha,$$

$$R = \frac{BD}{OB}.\ W = W\cos\alpha.$$

Ex. 2. *A string of length 7 feet carries a smooth ring of weight W so that the ring is free to slide on it, and has its two ends attached to two fixed points in a line 5 feet apart. If the ring is drawn aside by a horizontal force so that the two portions of the string become 3 feet and 4 feet, respectively, and one of them makes an angle of 60° with the horizontal, find the force and the tensions in the string.*

Let the fixed points be A and B and let the ring rest at O, so that $OA = 3'$ and $OB = 4'$. Since the sides of the triangle OAB are now 3, 4, 5, the triangle AOB is a right angled one.



If there were no horizontal force, the ring would be in equilibrium below the middle point of AB. But since the force F draws the ring towards A so that OA is reduced to 3 feet, it is OA which makes 60° with F. The angle which OB makes with the horizontal is, therefore, 30°.

The tensions in the two portions of the string are obviously the same.

Here four forces act at a point, and Lami's Theorem is not applicable.

Hence resolving the forces horizontally and vertically,

$$T\cos 30^\circ = T\cos 60^\circ + F. \qquad \ldots(1)$$

$$T\sin 30^\circ + T\sin 60^\circ = W. \qquad \ldots(2)$$

From (1), $T\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) = F. \qquad \ldots(3)$

and from (2), $T\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right) = W. \qquad \ldots(4)$

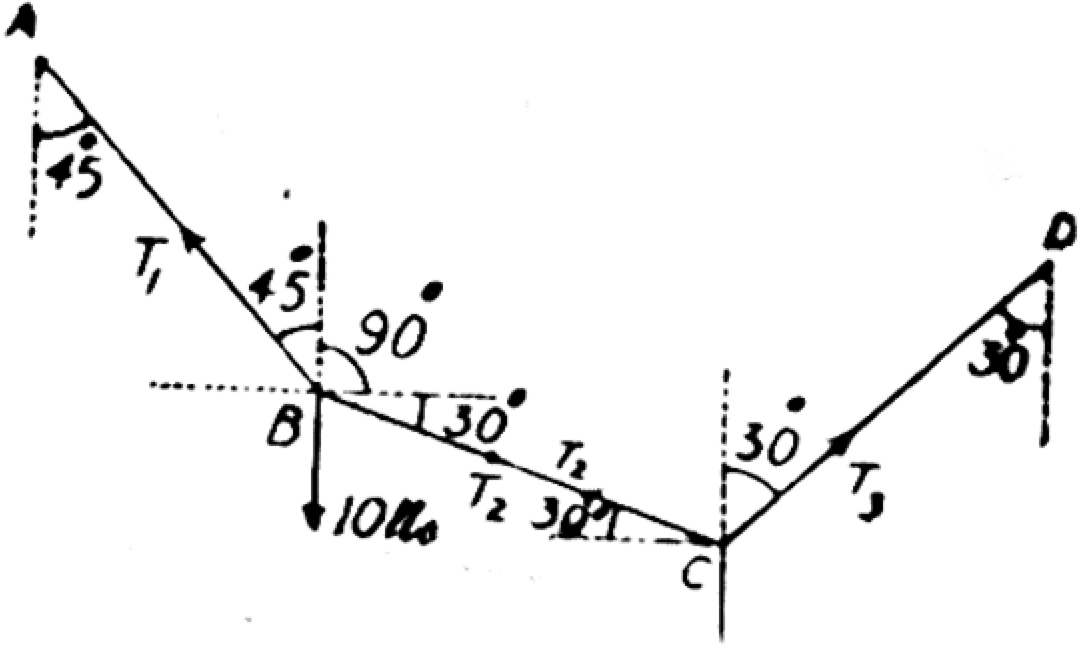
Hence, by division, $\frac{F}{W} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3}.$

$$\therefore \quad F = W(2-\sqrt{3}).$$

From (4) $T = \frac{2W}{\sqrt{3}+1} = (\sqrt{3}-1)W.$

Ex. 3. *ABCD is a thread suspended from points A and D and it carries a weight of 10 lbs. at B and a weight of W lbs. at C. The inclinations to the vertical of AB and CD are 45° and 30°, respectively, and the angle ABC is 165°. Find W and the tensions in the different parts of the thread.*

(*P. U. 1912*).



Let T_1 T_2, T_3 be the tensions in the parts AB, BC, CD of the thread. Then at point B act three forces T_1 along BA, T_2 along BC and 10 lbs. wt. vertically downwards. These forces keep B in equilibrium.

Hence by Lami's Theorem.

$$\frac{T_1}{\sin 60^\circ} = \frac{T_2}{\sin 45^\circ} = \frac{10}{\sin 165^\circ},$$

or, $T_1 = \frac{10}{\frac{\sqrt{3}-1}{2\sqrt{2}}} \times \frac{\sqrt{3}}{2} = \frac{10\sqrt{6}}{\sqrt{3}-1} = 5\sqrt{6}(\sqrt{3}+1)$ lbs. wt.

and $T_2 = 10(\sqrt{3}+1)$ lbs. wt.

Next we consider the equilibrium of the point C. The tension along CB at C is T_2, since the tension in the string CB is the same T_2 throughout. The point B is pulled along BC by T_2 and the point C is pulled along CB by T_2, and thus the string CB is in equilibrium. In fact, at every point of BC, two opposite tensions, each equal to T_2, act, and thus keep the point in equilibrium.

So here again by Lami's Theorem,

$$\frac{T_2}{\sin 30^\circ} = \frac{T_3}{\sin 60^\circ} = \frac{W}{\sin 90^\circ}$$

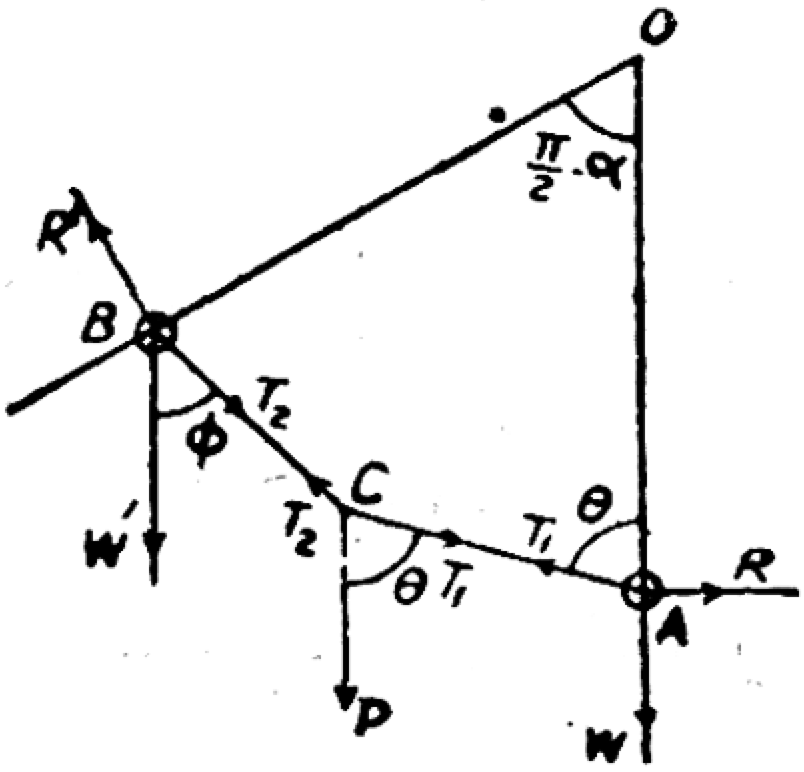
$\therefore \quad T_3 = \sqrt{3}.T_2 = 10(3+\sqrt{3})$ lbs. wt.,

and $W = 2T_2 = 20(\sqrt{3}+1)$ lbs. wt.

***Ex. 4.** *Two small smooth rings of weight W and W', connected by a string slide upon two fixed rods, the former of which is vertical, and the latter inclined at an angle α to the horizon. A weight P is tied to the string, prove that in the position of equilibrium,*

$\cot\theta : \cot\phi : \cot\alpha :: W : P+W : P+W+W'$ where θ, ϕ are the angles which the two portions of the string make with the vertical. (*P. U. 1917.*)

The above example, taken from Greaves' Statics, has an important bearing on tensions and reactions on particles.



The rings being smooth, the reactions at A and B are perpendicular to OA and OB.

Since equations involving reactions are not required, it would be convenient to resolve forces at each point along lines perpendicular to reactions.

Hence for pt. A, resolving along AO, $T_1 \cos\theta = W$ ......(1)

Similarly for B, resolving along BO,

$$T_2 \cos\left(\frac{\pi}{2}+\alpha-\phi\right) = W' \sin\alpha$$

or $\qquad T_2 \sin(\phi-\alpha) = W' \sin\alpha$ ......(2)

For considering the equilibrium of point C, Lami's Theorem is most convenient.

Hence

$$\frac{T_1}{\sin\phi}=\frac{T_2}{\sin\theta}=\frac{P}{\sin(\pi-\theta-\phi)}=\frac{P}{\sin(\theta-\phi)} \qquad \ldots\ldots(3)$$

Substituting for T_1 and T_2 in (3),

$$\frac{W}{\cos\theta\sin\phi}=\frac{W'\sin\alpha}{\sin\theta\,\sin(\phi-\alpha)}=\frac{P}{\sin(\theta-\phi)} \qquad \ldots\ldots(4)$$

Now equations (4) involve only the required quantities and are practically equivalent to the required result. Further reduction may be done as below :—

$$\because \quad \frac{W}{P}=\frac{\cos\theta\sin\phi}{\sin\theta\cos\phi-\cos\theta\sin\phi},$$

we have by componendo,

$$\frac{W}{P+W}=\frac{\cos\theta\sin\phi}{\sin\theta\cos\phi}=\frac{\cot\theta}{\cot\phi} \qquad \ldots\ldots(5)$$

Again, from (5), $W/\cos\theta\sin\phi$ in (4) can be replaced by $(W+P)/\sin\theta\cos\phi$, which leads to

$$\frac{W+P}{\cos\phi}=\frac{W'\sin\alpha}{\sin(\phi-\alpha)} \qquad \text{[a result free from } \theta\text{]}.$$

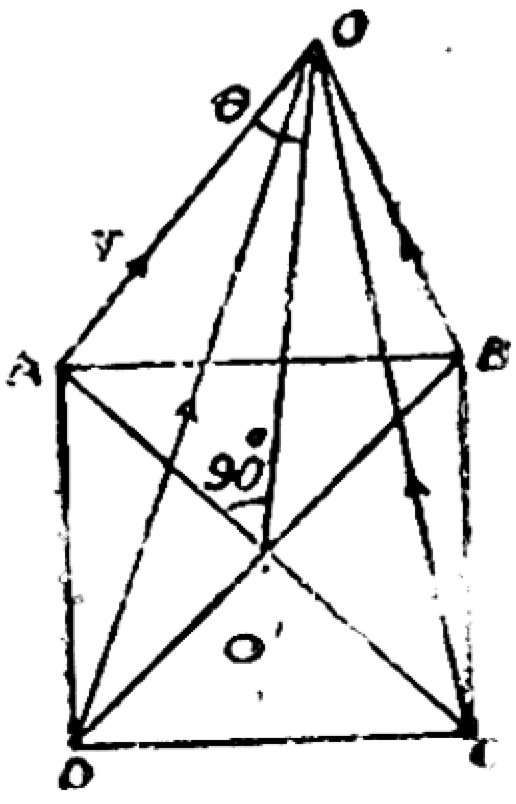
Hence $$\frac{W+P}{W'}=\frac{\sin\alpha\cos\phi}{\sin(\phi-\alpha)},$$

or $$\frac{W+P}{W+P+W'}=\frac{\sin\alpha\cos\phi}{\cos\alpha\sin\phi}=\frac{\cot\phi}{\cot\alpha}.$$

***Ex. 5** *A cradle in the form of a square box is suspended by means of four equal strings joined to one string attached to the ceiling. If the lengths of the strings be double the sides of the cradle, find the tension in each string in terms of the weight.*

[*P. U. 1929*]

Let ABCD be the upper face of the box, to the corners of which the strings are attached. Let T be the tension in any of the strings and *a* the side of the box. Since the cradle hangs symmetrically, the vertical through O, the point of suspension, meets the square at its centre O′, and OO′A is a right angled triangle.



Now $O'A = AB \cos 45° = \frac{a}{\sqrt{2}}$.

Also it is given that $OA = 2a$.

$\therefore$ If $\angle AOO' = \theta$, $\sin\theta = \frac{1}{2\sqrt{2}}$.

Hence $\cos\theta = \frac{\sqrt{7}}{2\sqrt{2}}$.

Resolving the forces vertically,

$4T\cos\theta = W$, where W is weight of the box.

$$\therefore \quad T = \frac{W}{4}\sec\theta = \frac{W}{4}\cdot\frac{2\sqrt{2}}{\sqrt{7}} = \frac{W}{\sqrt{14}}.$$

It is assumed here that the weight of the box is concentrated at a single point below OO'.

Exercises V

1. A weight of 10 lbs., which hangs by a string, is pushed to one side by a horizontal force so that the string makes an angle of 60° with the vertical; find the horizontal force and the tension of the string.

2. A weight W hangs by a string and is drawn aside by a horizontal force until the string makes an angle of 75° with the vertical; find the horizontal force and the tension of the string. Show also that the tension increases as the angle increases.

3. A weight W is suspended by two strings which make angles of 45° and 60°, respectively, with the vertical; find the tensions in the strings.

4. A given weight is suspended by two strings of lengths 5 ft. and 12 ft., respectively, and the other ends of the strings are attached to two fixed points in a horizontal line 13 ft. apart, find the ratio of the tensions in the strings, and the minimum strain which the strings should be able to stand.

5. A weight of 68 lbs. is suspended by two strings of lengths 8 ft. and 15 ft., respectively, and the other ends of the strings are attached to two fixed points in a horizontal line 17 ft. apart, find the tensions in the strings.

If the strings have any lengths so that they are at right angles, show that, for any given weight, the tension in the smaller string is always greater.

6. A string of length 30 ft. carries a ring of 1 lb. weight, which is free to slide on the string and the ends of the string are attached to two fixed points in a horizontal line 24 ft. apart. Find the tension in the string.

7. A mass of 12 lbs. placed on a smooth horizontal plane, is pulled by forces of 5 and $5\sqrt{3}$ lbs. wt., making angles 30° and 60°, respectively, with the horizontal plane, so that the mass remains in equilibrium. Explain how the forces will act and find the reaction of the plane.

8. A mass of M lbs. placed on a smooth horizontal plane is pulled by forces of P and Q lbs. wt., making angles 30° and 60° respectively, with the plane. The forces lie in the same vertical plane and on opposite sides of the normal. Find the conditions of equilibrium. Explain why the reaction cannot be negative.

9. A mass of 5 lbs. is placed on a plane inclined to the horizon at an angle of 30° and is prevented from sliding downwards by a nail driven in the plane. Find the pressures exerted by the mass on the nail and the plane.

10. A particle of weight W is placed on a smooth plane inclined to the horizon at an angle α. Find the horizontal force that would just prevent the particle from sliding downwards.

11. A mass of 100 lbs. rests on a smooth plane inclined to the horizon at an angle of 30° and is supported by a force whose inclination to the horizon is 60° ; find the force and the reaction of the plane.

12. Two smooth pegs in a wall are in a horizontal line and a string carrying two equal weights at its extremities passes over them. Find the reaction of each peg on the string and its direction.

13. There are three smooth pegs in a wall in the form of an equilateral triangle ABC, so that AC is horizontal and a string carrying two equal weights at its two ends passes over the pegs A, B, C in order. Find the reactions of the pegs on the string.

14 Two smooth pegs are fixed in a vertical wall in such a way that the straight line joining them makes an angle of 30° with the horizon. If a string passing over the pegs carries weights each equal to W at its ends, find the pressure on each of the pegs.

15. Find the greatest weight which can be supported by two light strings making angles 60° and 45° with the vertical, it being known that either string will break under a tension of W lbs. wt. (P. U. 1915)

16. A light string of length l has its ends fixed at A and B, two points in the same horizontal line, at a distance a apart, and a heavy ring of weight W can slide along the string. Prove that the ring can rest vertically beneath B, if a force $\frac{Wa}{l}$ is applied parallel to AB. (P. U. 1912)

17. A ring of weight W, which can slide freely on a smooth vertical circle, is supported by a string attached to the highest point. If the string subtends a right angle at the centre of the circle, find the tension of the string and the reaction of the circle on the ring.

18. If in the above example, the thread makes an angle θ with the vertical, find the tension and the reaction. (P. U. 1939)

19. Find the tensions in the string ABCD, fixed at A and D, and carrying equal weights at B and C, if AD (c) is horizontal, AB=BC=CD, and the length of the string is $3l$. (Ziwet and Field)

20. Two particles of weights P and Q, respectively are connected by a string which lies on a smooth circle fixed in a vertical plane; show that if $\frac{\pi}{2}$ be the angle subtended at the centre by the string, the inclination of the chord joining P, Q to the horizontal, in the position of equilibrium is

$$\tan^{-1} (P \sim Q)/(P+Q).$$ (Greaves)

21. A small weightless ring slides on a smooth arc of a circle and rests in equilibrium under the repulsion of three forces P, Q, R directed from points dividing the circumference into three equal parts; if its position of equilibrium lie on the smaller arc between the points from which the forces Q, R are directed, show that the pressure exerted by the circle is

$$\{P^2+Q^2+R^2-QR+RP+PQ\}^{\frac{1}{2}}.$$ (Ibid)

[*Hint.*—If the resultant force make an angle α with force P, resolve along the resultant force (which is along the normal) and the perpendicular direction. Eliminate α by squaring and adding the two sums of the components.]

22. A *heavy* chain has weights of 15 lb. and 20 lbs. attached to its ends and hangs in equilibrium over a smooth peg. Determine the weight of the chain if its greatest tension is 30 lb. wt.

23. A chain of length 10 ft. passing over a pulley has a weight of 9 lb. attached to one end which is at a distance of 4 ft. from the pulley. Find the weight of the chain.

24. ABC is an equilateral triangle formed of weightless inextensible strings. The side AB is horizontal, A and B are tied to fixed points D and E by equal weightless inextensible strings AD, BE, and a weight of 2 lbs. is attached at the lowest point C. Given that the angles DAB, ABE are each 150°, find the tension in each of the strings.

25. Three light equal strings are knotted together to form an equilateral triangle ABC, and a weight W is suspended from the lowest point A. If the triangle and the weight be supported with BC horizontal by means of two strings at B and C each at an angle of 135° with BC, show that the tension in BC is $\frac{W}{6}(3-\sqrt{3})$.

CHAPTER IV

RESULTANT OF FORCES ON A RIGID BODY

39. Like and Unlike Parallel Forces. We have hitherto considered the equilibrium of forces only when they act on a *particle*. To consider the action of forces on a *body*, the body must be taken to be *rigid*. (Art. 37). We have discussed that a force acting on a rigid body is completely known when we know its *magnitude*, *direction*, and *sense* ; the point of application is of no consequence, since *a force on a rigid body can be transmitted to act at any point in its line of action* (Art. 37).

When two non-parallel co-planar forces act on a body at two different points, their lines of action can be produced to intersect, and since their points of application can be transmitted to the point of intersection, their resultant can be found by the parallelogram law.

But when we have to compound two parallel forces acting on a body, we cannot construct a parallelogram, whose adjacent sides would represent the forces. This difficulty is overcome by the introduction of two equal and opposite forces which neutralize and do not produce any effect on the body. The method is shown in the following article.

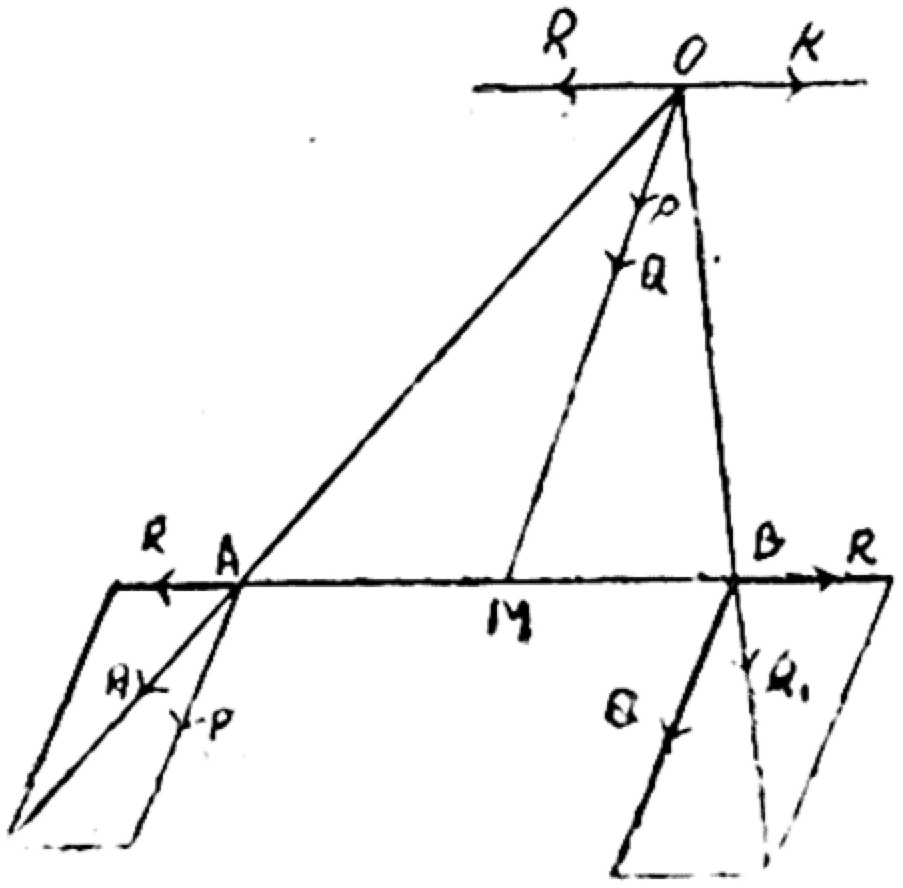
When parallel forces act in the same direction, they are said to be **like** ; when two forces act in opposite directions, they are called **unlike**.

40. Resultant of two parallel forces.

Case I. *When the forces are like.*

Let P and Q be two like parallel forces. Draw any line to meet their lines of action at A and B, and introduce at A and B, along BA and AB respectively, two forces each equal to R. The forces so introduced, being equal and opposite, have no effect on the system.

Let R_1, R_2 be the resultant of the forces at A and B, respectively. Produce them to meet at O and through O draw OM parallel to the forces, meeting AB at M.



Now the: resultant of P and Q is equivalent to the resultant of R_1 and R_2. But the point of application of R_1 may be transferred from A to O, and R_1 resolved into components R and P, parallel to their original directions. Similarly R_2 may be taken to act at O and resolved into Q and R. The two equal and opposite forces R at O cancel, and *we have finally the resultant $P+Q$ acting along OM.*

Having found the direction and magnitude of the resultant, *we proceed to determine the position of M* on AB.

Since R_1 is the resultant of **P** and R, R_1 acting in the opposite direction will be in equilibrium with P and R. These three forces acting at A, being in equilibrium, can be represented by the sides of the triangle OMA.

Hence
$$\frac{P}{R}=\frac{OM}{AM} \qquad \ldots\ldots(1)$$

Similarly, taking the forces at B,
$$\frac{Q}{R}=\frac{OM}{MB} \qquad \ldots\ldots(2)$$

From (1) and (2), by division,
$$\frac{P}{Q}=\frac{MB}{AM},$$

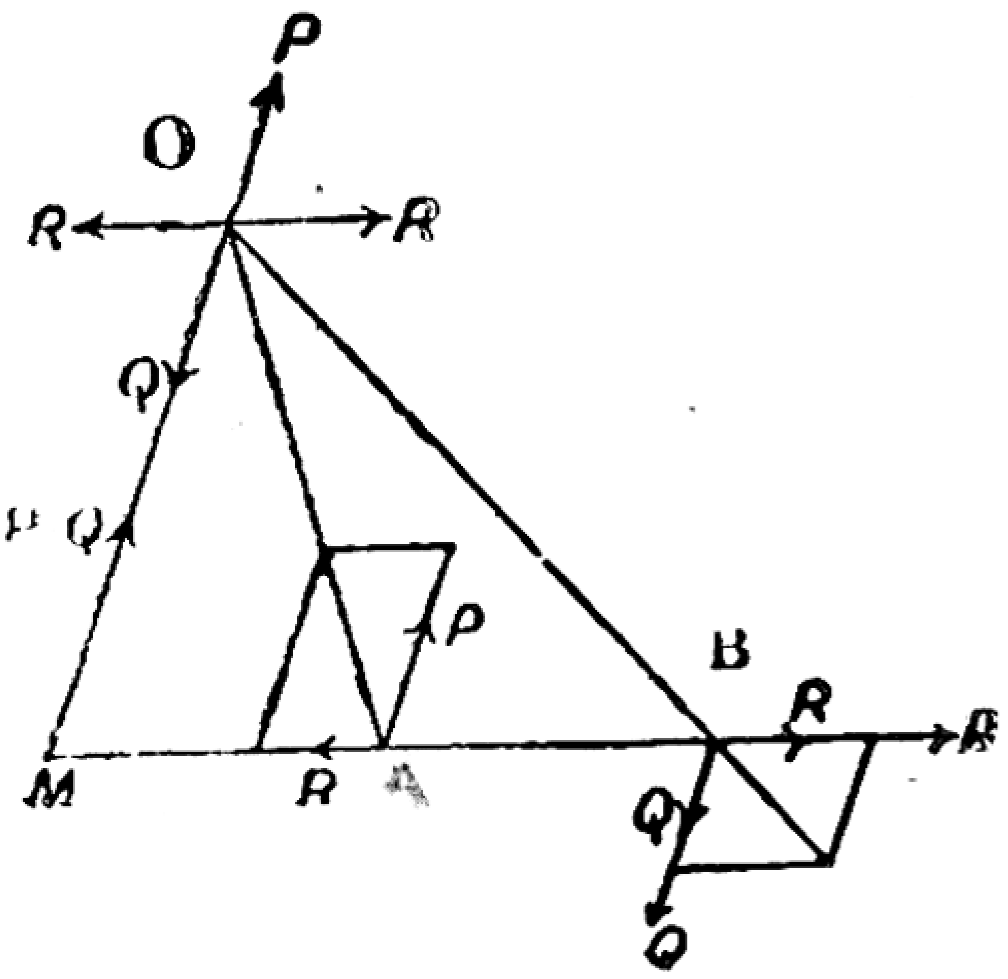
or $$P.AM=Q.MB,$$

i.e. M divides AB internally in the inverse ratio of the forces.

Case II. *When the forces are unlike.*

It will be seen that the above proof applies with slight necessary modification.

Let P and Q be two unlike parallel forces, P being the greater force. Draw any line to meet their lines of action at



A and B, and introduce at A and B, along BA and AB respectively, forces each equal to R. The forces so introduced have no effect on the equilibrium of the body. Let R_1, R_2 be the resultant of the forces at A and B, respectively. Produce them to meet at O, and through O draw OM parallel to the given forces to meet BA produced at M

Let the points of application of R_1 and R_2 be transferred to O Let R_1 be resolved into components P and R, parallel to their original directions. Similarly R_2 is resolved into components Q and R, each acting at O. Now the two equal and opposite forces R at O cancel and we have finally the *resultant force $P-Q$ acting along MO, i.e. parallel to the direction of the greater force.*

Now we have to *determine the position of M* on BA produced.

Since R_1 is the resultant of P and R, R_1 acting in the opposite direction will be in equilibrium with P and R. These three forces can therefore be represented by the sides of the triangle OMA.

$$\text{Thus} \quad \frac{P}{R}=\frac{OM}{AM} \qquad \text{......(1)}$$

Similarly taking the forces at B,

$$\frac{Q}{R}=\frac{OM}{BM} \qquad \text{......(2)}$$

$$\text{From (1) and (2),} \quad \frac{P}{Q}=\frac{BM}{AM}.$$

or, P.AM=Q.BM, *i.e.*, M divides BA *externally* in the inverse ratio of the forces.

Summary of results. When two like or unlike parallel forces P and Q act at points A and B of a rigid body, we summarise our conclusions as follows :

1. The magnitude of the resultant is the sum or difference of P and Q according as they are like or unlike.

2. The direction of the resultant is the same as that of P and Q when they are like ; but when P and Q are *unlike* the resultant has the same direction as that of the greater force.

3. The resultant is nearer the greater force and acts at a point M which divides the line AB or AB produced inversely as the forces, so that P.AM = Q.BM.

Note 1. The magnitude and direction of the resultant could have been found from elementary considerations. Since parallel forces may be regarded as a particular case of intersecting forces whose point of intersection is at infinity. The angle $\alpha = 0$ (Art. 17) for like forces, and equal to π for unlike forces. Then R becomes $P+Q$ and $P-Q$, respectively, and is obviously parallel to P, θ being zero.

Note 2. Since M divides AB externally in the inverse ratio of the forces, when they are unlike, the point M must be nearer to the greater force and on the other side of it.

Note 3. Supposing A and B to be the fixed position of M depends only upon the *magnitudes* of the forces P and Q. Then, whatever equal rotations P and Q may undergo, their resultant will be acting at the same point M, so long as their magnitudes have the same proportion.

Note4. The *construction in Case II fails when R_1 and R_2 become parallel, i.e.*, when they make the same angle with AB. *This happens when P and Q become equal in magnitude.* Thus we conclude that *two parallel equal and opposite forces cannot have any single force which would have the same effect on the system as these forces.* So it will be necessary to consider this exceptional case, separately.

Note 5. Taking regard of the signs of the forces, if they are like, they will have the same signs ; if unlike, opposite signs. Also when the forces are like, AM, MB will be measured in the same direction, but if unlike, in opposite directions. Hence paying regard to signs, the formula.

$$P.AM = Q.MB \text{ or } AM/MB = Q/P$$

gives the point M *for both the cases.*

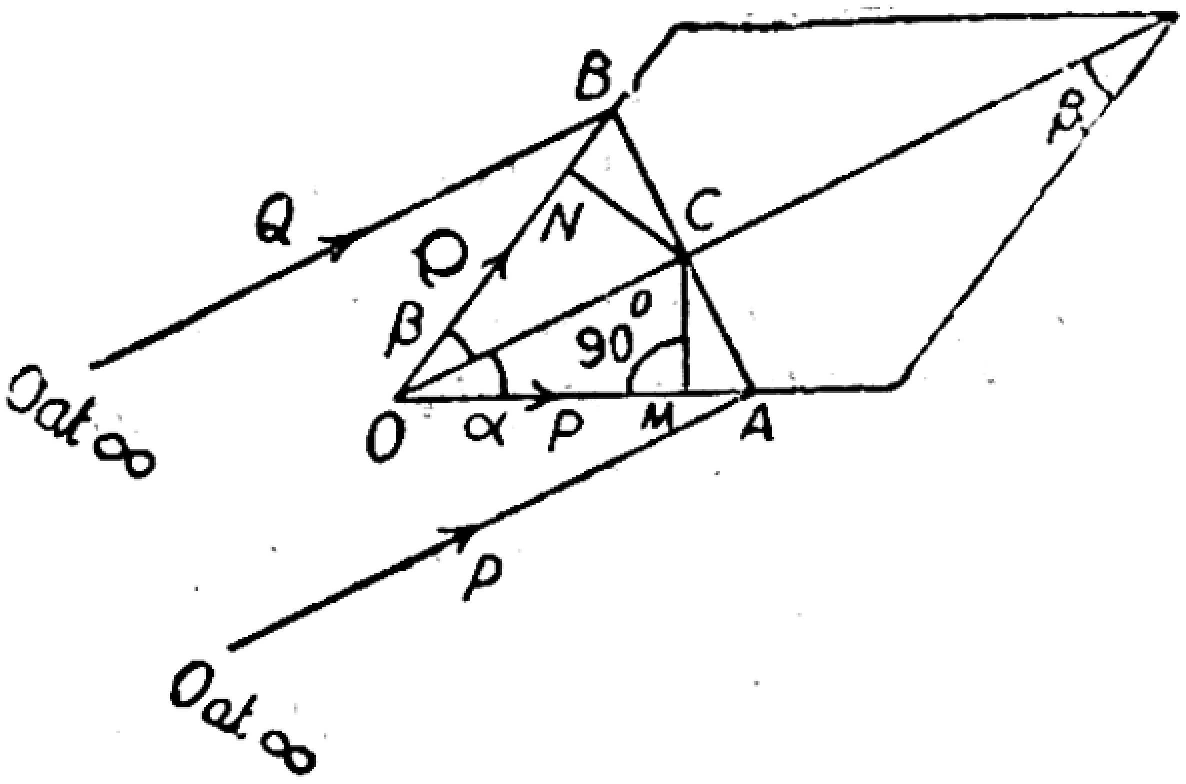
Note 6. This proposition enunciates the Principle of the Lever. If a horizontal bar AB rests on a support at M and weight P is placed at the point A (say), then the force Q at B, which would keep the system equilibrium, must act perpendicular to AB, and be such that

$$P.AM = Q\,MB.$$

M is called the *fulcrum*, P the *weight*, and Q the *power*.

This principle, as stated already, (Introduction, Art. 3) *was until recently regarded as the fundamental principle in Statics.*

41*. It is interesting to note that the *position* also of the resultant of two like parallel forces could be deduced from the parallelogram of forces.



We have

$$\frac{P}{Q}=\frac{\sin\beta}{\sin\alpha}=\frac{OC\sin\beta}{OC\sin\alpha},$$

where C is any point on the resultant.

If CM, CN are perpendiculars on P and Q, the above becomes

$$\frac{P}{Q}=\frac{CN}{CM} \qquad \text{......(1)}$$

Now take the point O to infinity, the forces become parallel and CM, CN take the positions of CA and CB.

Hence, (1) gives $P/Q = BC/CA$.

42. **The resultant of any number of parallel forces.** This principle of compounding two parallel forces (unequal, when they are unlike) is easily capable of extension, so that it will apply for compounding any number of parallel forces. The resultant R_1 of two forces P and Q being found, the resultant R_2, of R_1 and a third force may then be found; and the process continued until the final resultant is reached; care should be taken to compound with the resultant at any stage a force which is *not equal and unlike* to it. Ultimately, *either a single force will be obtained as the resultant, or two equal and unlike forces*; the resultant in the former case will be the algebraic sum of the forces.

43. **The centre of gravity of a body.** The weight of a body being the force with which it is attracted to the Earth, it is obviously the resultant of the gravitational forces acting on the particles of which the body is composed. Since the bodies which we deal with on this Earth, are small in comparison with the distance of the centre of the Earth from its surface, the forces on the particles may, for all practical purposes, be taken to be parallel. These forces may then be compounded

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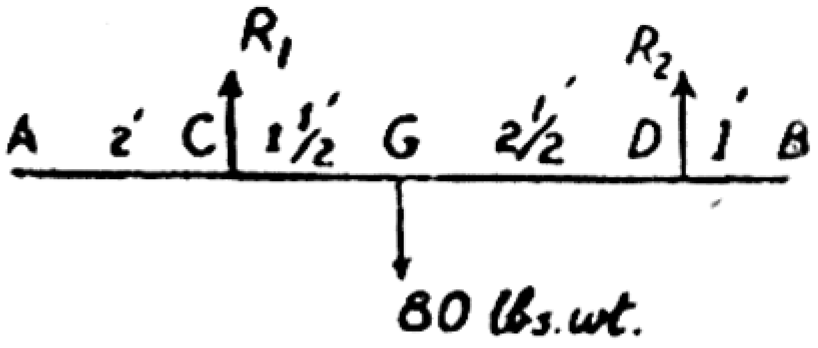
*This Proof is due to Principal G. L. Gupta.

by the rule of compounding parallel forces and the point at which the total weight would act is called the *centre of gravity* of the body. It will be proved later on that this point is a fixed point in the body, in whatever position the body be placed.

We shall *assume for the present that the centre of gravity of a uniform rod, or beam, is at the middle of its length,* situated symmetrically with respect to its thickness. If the thickness be neglected, the centre of gravity would merely be the middle point of the length of the rod.

44. Ex. 1. *A uniform beam, 7 feet long, is supported in a horizontal position by two props which are 4 feet apart, so that the beam projects 1 foot beyond one of the props. If the beam weigh 80 lbs. find the pressures on the two props.*



The weight acts at the middle point G of the length of the beam, and if the props are at C and D, CG=1½' and GD=2½'.

Let R_1, R_2, be the reactions of the props. These reactions being in equilibrium with the weight at G, we have the equations :—

$$R_1+R_2=80, \qquad \ldots\ldots(1)$$

$$\tfrac{5}{2}R_2=\tfrac{3}{2}R_1. \qquad \ldots\ldots(2)$$

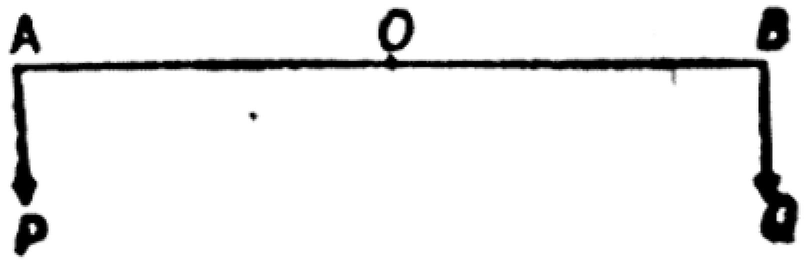
Solving (1) and (2),

$R_1=50$ lbs, wt., and $R_2=80$ lbs. wt.

Ex. 2. *The resultant of two like parallel forces P, Q passes through a point O. When P is increased by R and Q by S, the resultant still passes through O; and also when Q, R replace P, Q respectively. Prove that*

$$S=R-\frac{(Q-R)^2}{P-Q}. \qquad \textit{(P. U. 1928)}$$

Let P, Q act at A, B, respectively in the line AOB.



Then $P.OA=Q.OB.$ ......(1)

Also $(P+R)OA=(Q+S)OB,$ ......(2)

and $Q.OA=R.OB.$ ......(3)

Now, by division, from (1) and (2),

$$\frac{P+R}{P}=\frac{Q+S}{Q}, \text{ or } \frac{R}{P}=\frac{S}{Q}. \qquad \ldots\ldots(4)$$

Also from (1) and (3),

$$\frac{P}{Q}=\frac{Q}{R} \qquad \ldots\ldots(5)$$

If our object were merely to find the value of S, we have it from (4), and the condition (3) would be unnecessary.

In order to find S in the required form, we see from (5)

$$\frac{P-Q}{Q}=\frac{Q-R}{R}, \text{ or } \frac{R}{Q}=\frac{Q-R}{P-Q}.$$

Hence

$$R-\frac{(Q-R)^2}{P-Q}=R-\frac{(Q-R)R}{Q}=\frac{R^2}{Q}. \qquad \ldots\ldots(6)$$

Multiplying (4) and (5), $\frac{R}{Q}=\frac{S}{R}$, or $S=\frac{R^2}{Q}$.

$$\therefore \quad S=R-\frac{(Q-R)^2}{P-Q}. \qquad \text{[from (6)}$$

Exercises VI

1. Find the resultant of each of the following pairs of like parallel forces, determining its distances from each of the component forces :—

(*i*) 2 lbs. and 4 lbs , 1 ft. apart ;
(*ii*) 5 lbs. and 7 lbs., 2 ft. apart ;
(*iii*) 10·25 lbs. and 13·75 lbs. 18 inches apart ;
(*iv*) 45 and 35 grammes, ·25 of a metre apart.

2. If the forces of each pair in Example 1 be unlike, find the resultant and its distances from the components, giving the figures in each case.

3 A rod 7 ft. long rests on a pivot 3 ft. distant from one end by which hangs a weight of 5 lbs. Find the weight to be attached to the other end so that the rod may be in equilibrium and find the pressure on the pivot.

4. A uniform beam of weight 40 lbs. and length 8 ft. is carried on the shoulders of two men A and B, so that it projects 6 in. over the shoulder of A and 1 foot over the shoulder of B. If a weight of 51 lbs. is suspended at the middle point of the length of the beam, find the pressure exerted on the shoulder of each.

5. Two boys of weight 50 lbs. and 60 lbs., sit on a plank 7 ft. long, which rests on a support dividing the plank in the ratio 2 : 3. If the lighter boy sits at the end of the shorter portion, find the position of the other boy so that the plank may balance ; the weight of the plank is neglected.

6. One arm of a balance, 17 in. long, is 1 in. longer than the other. If the goods to be weighed are suspended from the longer arm, find the real weight when the apparent weight is 36 lbs., the weight of the beam of the balance being supposed to act at the point of suspension.

7. Two unlike parallel forces P and Q act at points 5 ft. apart ; if the resultant force be 9 lbs. wt. and act a a distance of 10 ft. from P the greater force, find P and Q.

8. Two unlike parallel forces, the greater of which is 72 lbs. wt., have a resultant 24 lbs. wt. Find the ratio of the distances of the resultant from the component forces.

9. A uniform rod 1·5 metres long and weighing 2·25 kilogrammes is laid on a table with 25 cms. projecting ove the edge. What weight can be hung on the end of the rod before it will be pulled over.

10. It is required to hang a weight of a tons from the middle point of a girder, one end of which rests on a support which can just bear a weight of b tons $\left(b < \frac{a}{2}\right)$. Neglecting the weight of the girder, find the farthest distance from the middle point, at which the other support can be placed.

(*P. U. 1922*).

11. A uniform beam 8 ft. long and weighing 20 lbs., placed on a prop. and a string tied to the farther end, holds it in the horizontal position. If the string can bear a maximum tension of 8 lbs., find the least distance of the prop from the other end.

12. The resultant of two unlike parallel forces P and Q (P > Q), acts at a certain point O. If the forces be both increased by R, show that the perpendicular distance of O from P increases.

What happens if the magnitude of R be increased beyond any limit ?

13. If three parallel forces are in equilibrium, each force is proportional to the distance between the other two.

Hint : We have

(*i*) $\frac{P}{MB}=\frac{Q}{AM}=\frac{P+Q}{AM+MB}=\frac{R}{AB}$, when the forces are like.

(*ii*) $\frac{P}{BM}=\frac{Q}{AM}=\frac{P-Q}{BM-AM}=\frac{R}{AB}$, when the forces are unlike.

14. P and Q are like parallel forces. If Q be moved parallel to itself through a distance x, prove that the resultant of P and Q moves through a distance $\frac{Qx}{P+Q}$.

15. Two like parallel forces P and Q act at given points of a body ; if Q be changed to $\frac{P^2}{Q}$, show that the line of action of the resultant is the same as it would be if the forces were simply interchanged.

16. Three equal like parallel forces act at the middle points of the sides of a triangle. Show that their resultant passes through the point of intersection of the medians of the triangle.

17. Four forces are represented by AB, 2BC, 3CD and 4DA, where ABCD is a given *square*, show that the resultant will be represented in magnitude and direction by 2CA and show its position.

18. A man carries a bundle at the end of a stick which is placed over his shoulder. If W be the weight of the bundle and a, x the distances of the bundle and his hand from the shoulder, show that the pressure on the shoulder is $W\left(1+\frac{a}{x}\right)$. If the distance between his hand and his shoulder be changed, how does the pressure on his shoulder change ?

19. A uniform rod whose weight is 8 lbs. is placed upon two props which are in the same horizontal line and 6 inches apart. Find the distance to which the ends of the rod extend beyond the props, if the difference of the pressures on the props is 4lbs., and the length of the rod 8 feet.

20. A light rod AB 8 feet in length is kept in a horizontal position by two vertical strings attached at the ends A and B. Two equal weights P are attached at distances 8 inches from A and 12 inches from B. If the *maximum tension* that either string can sustain is 50 lbs. find the greatest value of P.

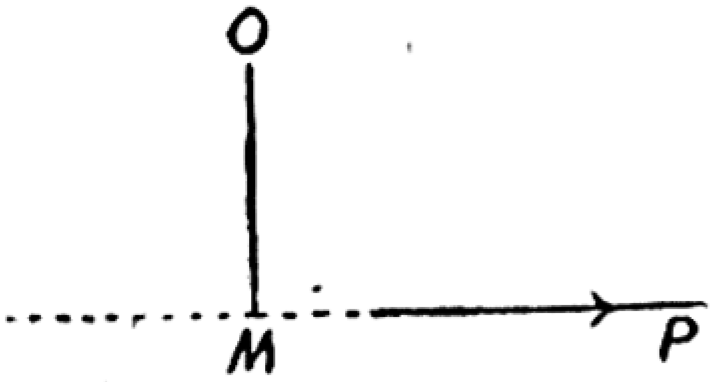
21. Break up a force P into two like parallel forces in the ratio $m : n$; if one acts at a distance a from P, find the distance at which the other force will act from P.

22. A rod 14 feet long without weight has a weight of 4 lbs. suspended from its middle point. The rod can turn about one end. If the rod is to be sustained by a vertical force of 11 lbs. weight at one end, where must an additional weight of 63 lbs. be attached, in order that the rod may remain at *rest* ?

Moments

45. **Definition.**—*The* **moment** *of a force about a point is the product of the force and the perpendicular from the point on the line of action of the force.*

Thus the moment of a force P about any point O is P×OM, where OM is the perpendicular on the line of action of P.



Since the product increases with the increase in magnitude of the force, or in length of the perpendicular, the moment may be taken to measure *the tendency of the force to turn the body about the point.*

It is evident that the moment of a force is zero when the perpendicular OM vanishes, *i. e.* when the line of action of the force passes through the point.

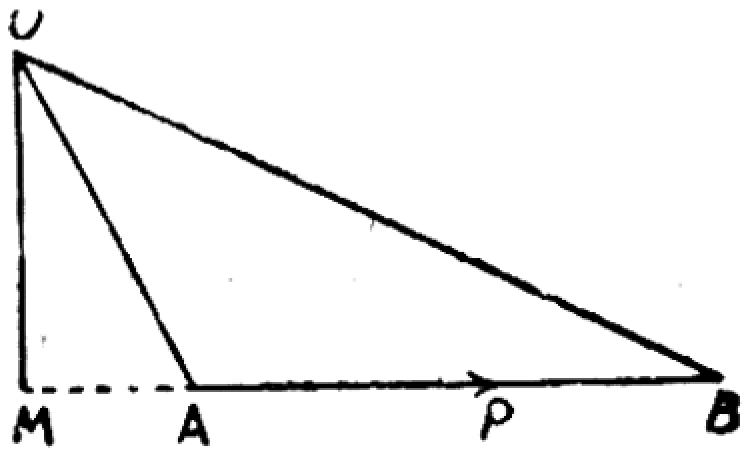
In a common balance with equal arms, out of the two masses placed in the two scale pans, the weight of the greater mass turns the arm, since its moment about the pivot is greater. If masses were equal but arms unequal, the greater arm, giving the greater moment, will turn downwards.

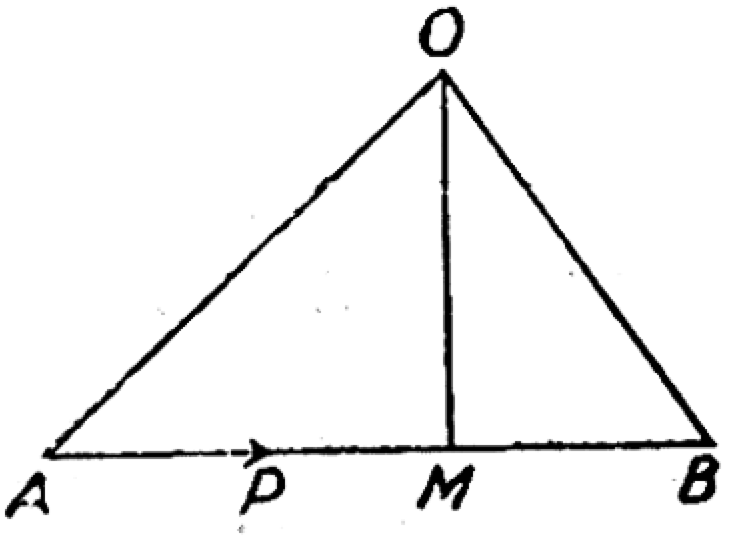
Sign of the moment : Convention.—Since one force may have the turning tendency in a direction opposite to that of another force, it is obvious that a moment in addition to its *magnitude* will have a *sign* also. If a force has a tendency to turn a body about a point in the counter-clockwise direction, its moment about the point is to be taken as **positive**; if in the clockwise direction, its moment is **negative**.

Note 1.—Where there are several forces, the sum of their moments about any point is their *algebraic sum.*

Note 2.—The unit of moment is usually taken to be the moment of a force of one lb. wt. about a point one foot distant from it.

45. Geometrical representation. If a force P acts along, and is represented by AB, then its moment about O is represented by AB×OM, where OM is perpendicular on AB or AB produced.





Since AB×OM=2×area of △ OAB, *the moment is gometrically represented by twice the area of the triangle formed by the force as base and the point about which the moment is taken as vertex.*

Ex: 1. Forces of 2 lbs., 4 lbs., and 8 lbs., act along the sides AB, CB, CD, AD, respectively, of a square ABCD, whose side is 4 ft. Find the algebraic sum of the moments of the forces about the centre. − 12 [illegible]

Ex. 2. Equal forces act along the sides of an equilateral triangle, taken one way round; show both *geometrically* and *analytically* that the algebraic sum of the moments of the forces about the centroid is equal to the moment of the forces about the opposite vertex.

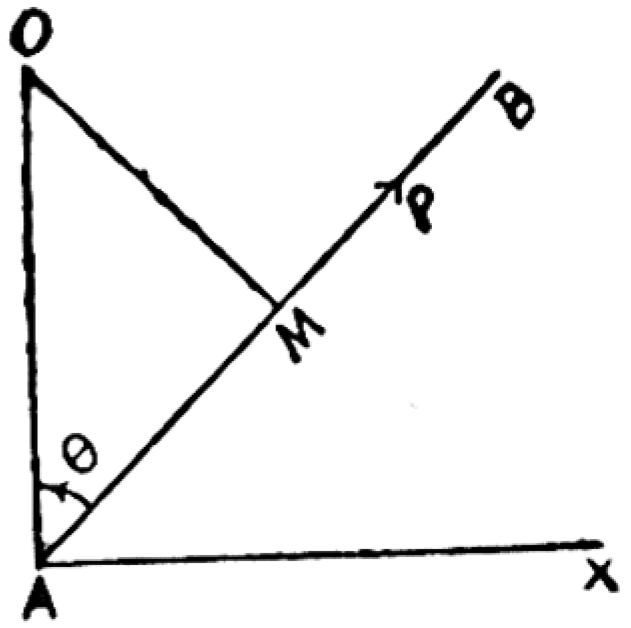
[*Hint*: The perp. from the centroid on any side is one-third of the perp. upon it from the opposite vertex.

Ex. 3. A heavy uniform rod AB is suspended from its middle point M; if forces of 1, 2, 3 lbs. wt. act at distances of 1, 2, 3, ft. respectively, from M in the portion MB, and forces of 4, 5, 6 lbs. wt. act at distances of 2, 3, 4 ft., respectively, from M in the portion AM, find the algebraic sum of the moments of the forces about M. 33 lb.wt.

46. We shall deduce another expression for the moment of a force about a point.

Let the force P be represented by AB, and let O be the point about which the moment is to be taken.

Join OA and draw AX perpendicular to OA. Then the moment of the force about O



$=P\times OM$
$=P\times OA\sin\theta\ [\angle MAO=\theta]$
$=OA\times P\sin\theta$
$=OA\times$ resolved part of P perpendicular to OA.
(*i. e.* OA × resolved part of P along AX).

Note. We have thus obtained *three expressions* for the moment of a force P (represented by AB) about a point O. They are

(*i*) $P\times OM$; (*ii*) 2 times area of $\triangle$ OAB; (*iii*) $OA\times P\sin\theta$ (or OA × resolved part of P perpendicular to OA).

Corollary. Since $P\sin\theta$ acts along AX which is $\perp$ to OA, we have

Moment of $P\sin\theta$ about $O=P\sin\theta\times OA$.
$=$ moment of P about O.

The corollary may be stated in words as follows :

The moment of a force about a given point is algebraically equal to the moment of its resolved part at right angles to the line joining the point with the particale on which the force acts."

47. Varignon's Theorem of Moments.—*The algebraic sum* **of the moments of two forces about any point in their plane is equal to the moment of the resultant about that point.**

Case I.—*When the forces intersect.*

Let the two forces P, Q acting along AB, AC have a resultant R. Let O be the point about which the moments are to be taken.

Draw OD parallel to the direction of P, cutting the direction of Q at C. *Take length AC to represent Q in magnitude, and on the same scale. Let AB represent P.*

Complete the $\|^{m}$ ABDC. Then AD represents R.

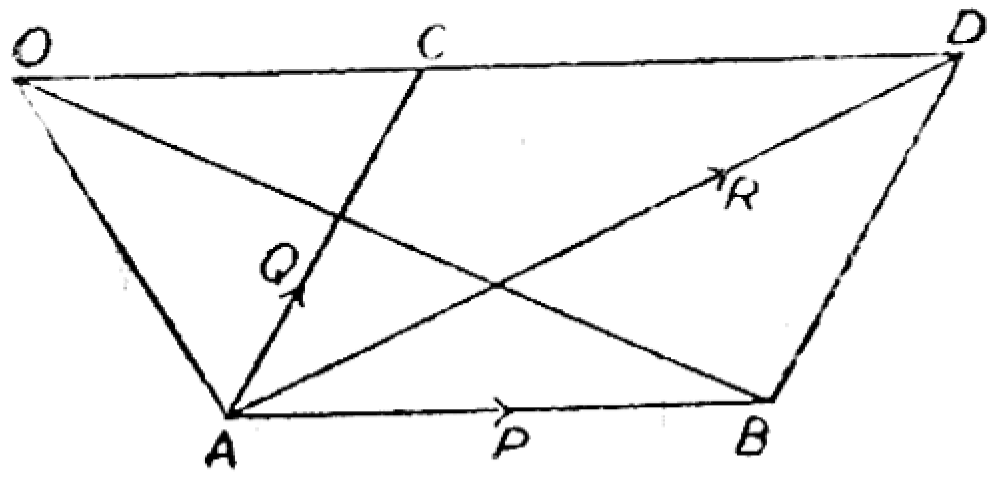


Fig. I

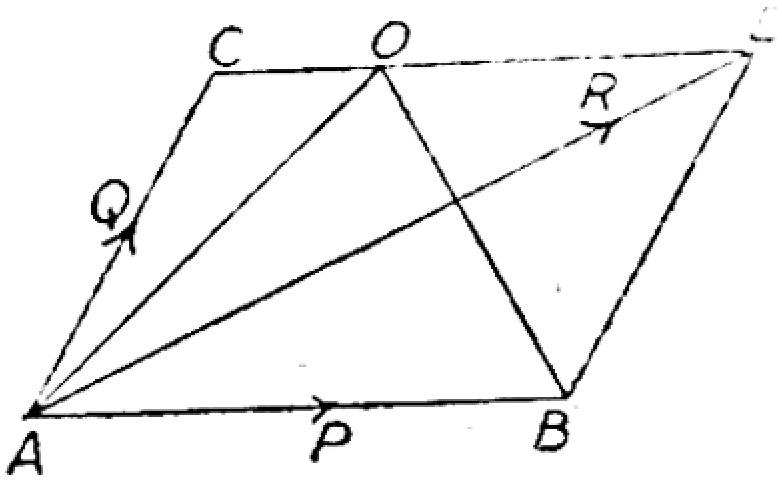


Fig. II

Join OA, OB.

(*a*) *When the point lies outside* $\angle CAB$
To prove Mom. of P about O + Mom. of Q about O.
= Moment of R about O.

i. e. to prove (*i*) $2\triangle OAB + 2\triangle OAC = 2\triangle OAD$ in Fig. 1.

Now Since OD is $\|$ to AB, we have $\triangle OAB = \triangle ABD = \frac{1}{2}\|^{m}$ ABDC $= \triangle ACD$.

In Fig I, adding $\triangle OAC$ to both sides, we have $\triangle OAB + \triangle OAC = \triangle ACD + \triangle OAC = \triangle OAD$

Hence, $2\triangle OAB + 2\triangle OAC = 2\triangle OAD$.

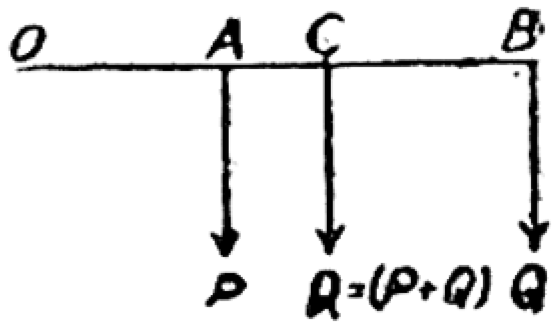
(*b*) *When the point O lies inside* $\angle CAB$
We have $\triangle OAB = \triangle ACD$ (*already proved*)
In Fig 2, Subtracting $\triangle OAC$ from both sides we have
$\triangle OAB - \triangle OAC = \triangle ACD - \triangle OAC = \triangle OAD$
$\therefore 2\triangle OAB - 2\triangle OAC = 2\triangle OAD$.

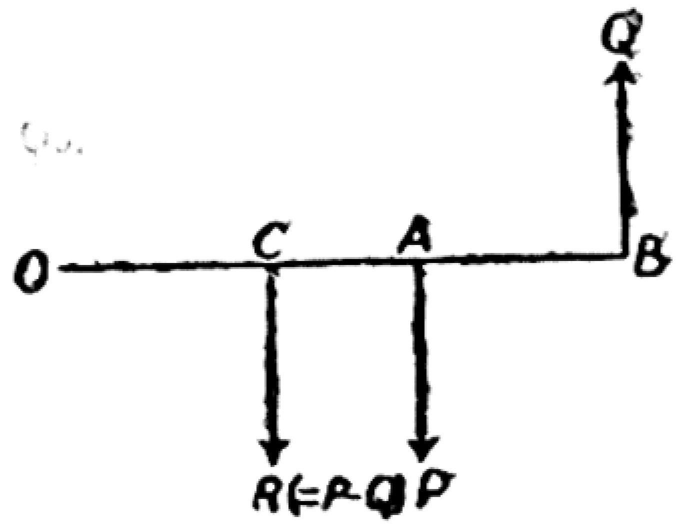
Observe that in Fig II, the moment of P is positive but the moment of Q is negative, whereas in Fig I, both the moments are positive.

Case II. *When the forces are parallel.*

Let P, Q be the two forces, *like or unlike* and O any point in the plane of the forces about which moments are to be taken.

Draw a perpendicular from O to the directions of the forces, meeting P, Q and their resultant R $(= P + Q)$ in A, B, C respectively.





Now, by Art 26, P, AC = Q, CB ....(1)

In Fig. I,

The algebraic sum of the moments of P and Q about O

$= P.\ OA + Q,\ OB$

$= P\ (OC - AC) + Q\ (OC + CB)$

$= (P+Q)\ OC - P.AC + Q.\ CB$

$= (P+Q)\ OB\ [\because -P.\ AC + Q.\ CB = 0$ by (1)

$= R.\ OC$

= Moment of R about O.

In Fig. II, the moment of Q is of opposite sign to the moments of P and R and we have, (P being > Q), algebraic sum of the moment of P and Q about O.

$= P.\ OA - Q.\ OB$

$= P\ (OC + CA) - Q\ (OC + CB)$

$= (P-Q)\ OC + P.\ CA - Q.\ CB.$

$= (P-Q)\ \ OC\ \ [\because P.CA - Q.\ CB = 0$ by (1)]

$= R,\ OC$

= Moment of R about O.

Note. 1. Taking regard of the signs of the forces P, Q and lengths AC and CB, we note that when P, Q are like, AC, CB will have the same signs ; and when they are unlike opposite signs.

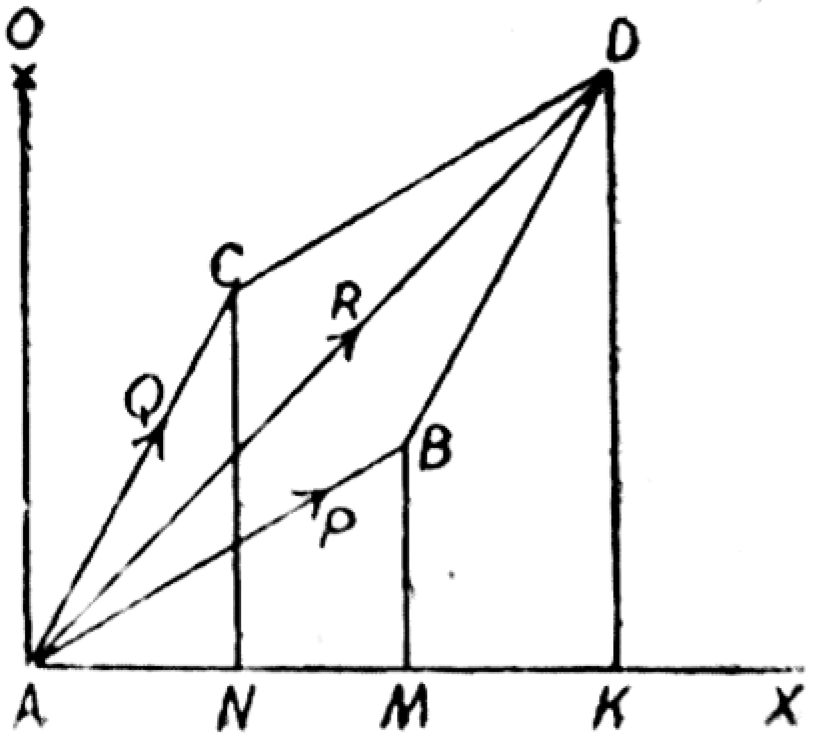
Note 2. The proposition may be easily established when O is between P and Q or when it has any other position.

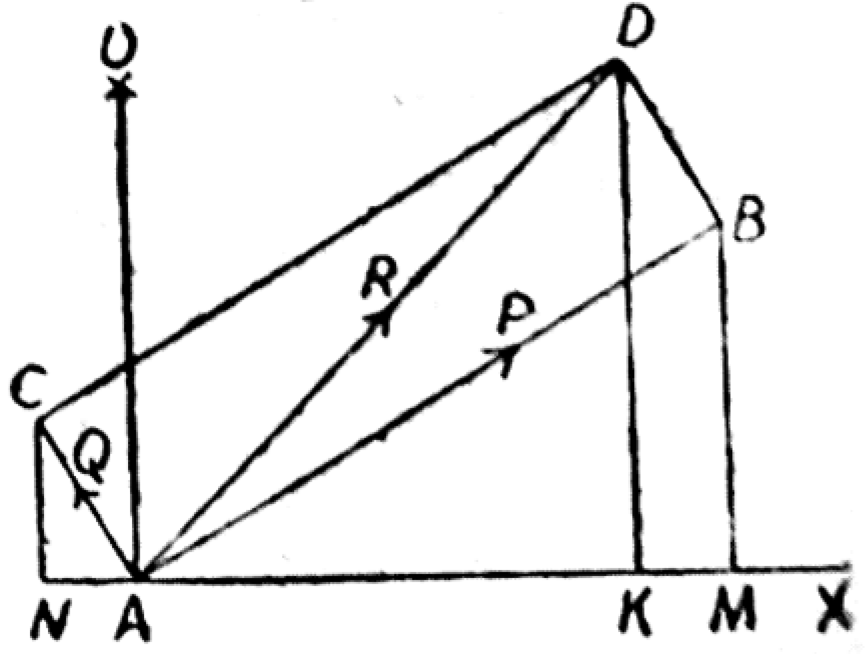
Alternative Proof of Varignon's Theorem case I.

Let the forces P, Q act at A and their resultant R be represented by the lines AB, AC, AD of the ||^m ABDC; and let the moment be taken about O.

Join OA and *draw AX perpendicular to OA in the* plane of the forces.

To prove Mom. of P about O + Mom. of Q about O
= Mom. of R about O





Draw BM, CN, DK perpendiculars on AX.

Now,

(1) Mom. of P about O$=$Mom. of AB about O$=$OA$\times$ resolved part of AB along AX [Art. 45]

$$=OA \times AM$$

Similarly

(2) Mom. of Q about O$=$Mom. of AC about O$=$OA$\times$AN

Adding (1) and 2, we get

Mom. of P about O$+$Mom. of Q about O

$$=OA \times (AM+AN)=OA \times AK$$

$[\because AM+AN=AK$ Art. 30]

$=$Mom. of AD about O.

$=$Mom. of R about O.

In Fig II, Mom. of Q about O$=$Mom. of AC about O

$$=-OA \times AN.$$

and we get

Mom. of P about O$-$Mom. of Q about O

$=$Moment of R about O,

Showing that in both cases, *algebraic sum* **of Moment of the forces$=$Moment of the resultant.**

48. **General theorem of moments.**—*The algebraic sum of the moments of any number of coplanar forces about a point is equal to the moment of the resultant about that point.*

By proceeding in a manner similar to that in Art. 31, Varignon's theorem can easily be extended to give the general theorem of moments. The proof is left to the student.

Cor 1. *It follows from this theorem that if the algebraic sum of the moments, of the forces about a point in the plane vanishes, either* (*i*) *the point lies on the resultant, or* (*ii*) *the resultant vanishes, and the forces are in equilibrium.*

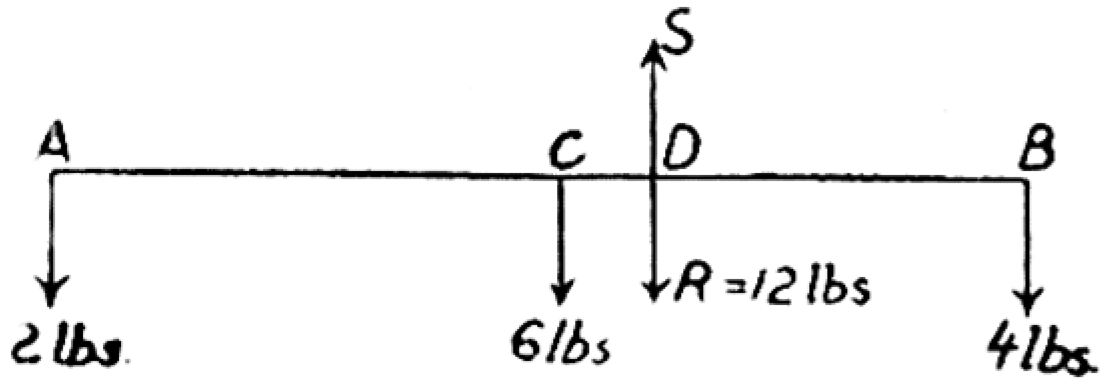
This fact is very useful in finding the line of action of the resultant. If the forces are parallel, we have only to find *one* such point about which the algebraic sum of the moments of the force would vanish. In the case of *non-parallel forces, two points* will determine the position of the resultant.

The magnitude and direction of the resultant being determined by the application of Art. 32 for non-paralle forces, the resultant will thus be completely determined whether the forces are concurrent or not.

Cor. 2. If the sum of the moments of co-planar forces about two points be the same, the resultant is parallel to the straight line joining the points.

49. Ex. 1. *A uniform rod 3′ long, and weighing 6 lbs. has weights of 2 and 4 lbs, respectively attached to its ends. Find the point about which the rod would be balanced.*

The rod would balance about the point where the resultant meets A B. The position of the resultant could be found by compounding *first* 2 lbs. and 4 lbs. and *then* compounding the force of 6 lbs. so obtained with the remaining force of 6 lbs But the Principle of Moments would get the result more easily.



I Method. Let the final resultant R (=12 lbs.) act at D. Since the algebraic sum of the moment of the forces about B is equal to the moment of the resultant about B,

$$2.3+6.\tfrac{3}{2}+4.0=12.\text{DB}$$

$$\therefore \quad \text{DB}=\tfrac{15}{12}=1'\ 8''.$$

If a prop is placed at D, the rod would balance and the reaction S=12 lbs.

II. Method. Let DB$=x$, then CD$=\frac{3}{2}-x$.

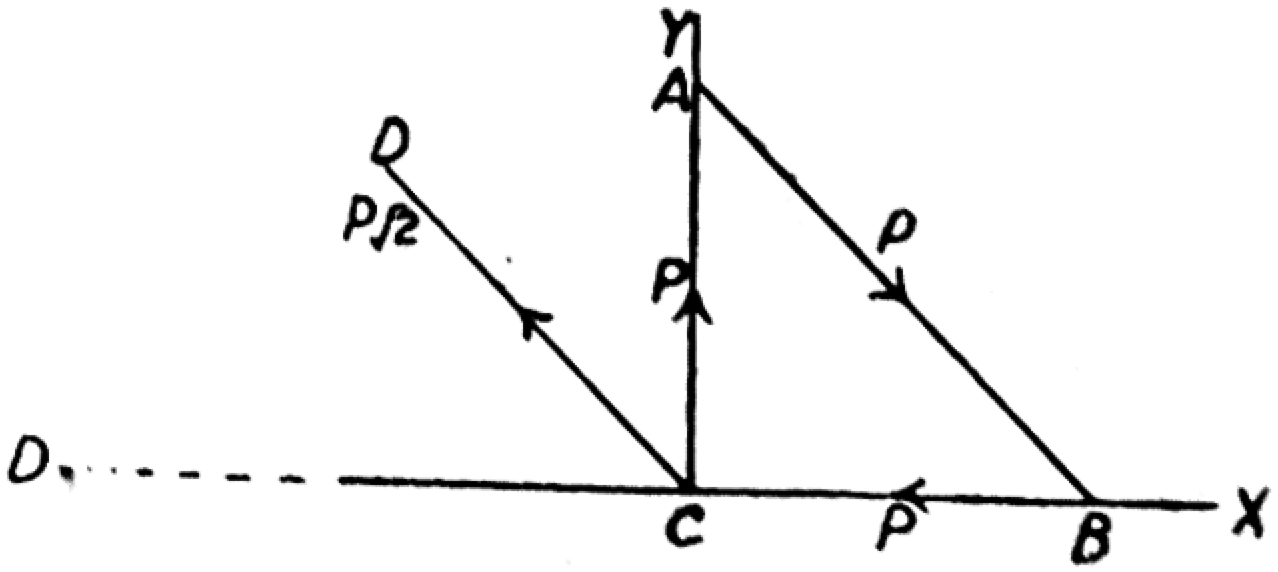
Since the algebraic sum of the moments of the forces would vanish about D,

$$2(3-x)+6(\tfrac{3}{2}-x)=4x,$$

which equation gives the value of x.

Ex. 2. *ABC is an isosceles triangle, right-angled at C, three equal forces act along AB, BC, CA. Find the resultant completely.* [*P. U. 1906*]

I Method. The resultant of forces along CA and BC is P$\sqrt{2}$ along CD, where CD is parallel to BA. Thus the final resultant of parallel forces P$\sqrt{2}$ and P along AB may be found. The magnitude is P$(\sqrt{2}-1)$ and direction parallel to BA.



II Method. It is obvious that the *magnitude and direction* of the resultant would be the same as if the forces were concurrent at any point (say C). Taking CB and CA for coordinate axes, we have

$$X = -P + P \cos 45° = -\frac{P\sqrt{2}-1}{\sqrt{2}},$$

$$Y = P - P \sin 45° = \frac{P\sqrt{2}-1}{\sqrt{2}}.$$

Hence R (resultant) $= \sqrt{X^2+Y^2} = P(\sqrt{2}-1)$.

f R makes an angle θ with OX, $\tan\theta = \frac{Y}{X} = -1$

$\therefore$ $\theta = 135°$, *i.e.*, R is parallel to BA.

The line of action of R may be found by determining the point where R would meet BC. Let it be D and let $CD = x$, and $CB = a$. Taking moments about D, we have

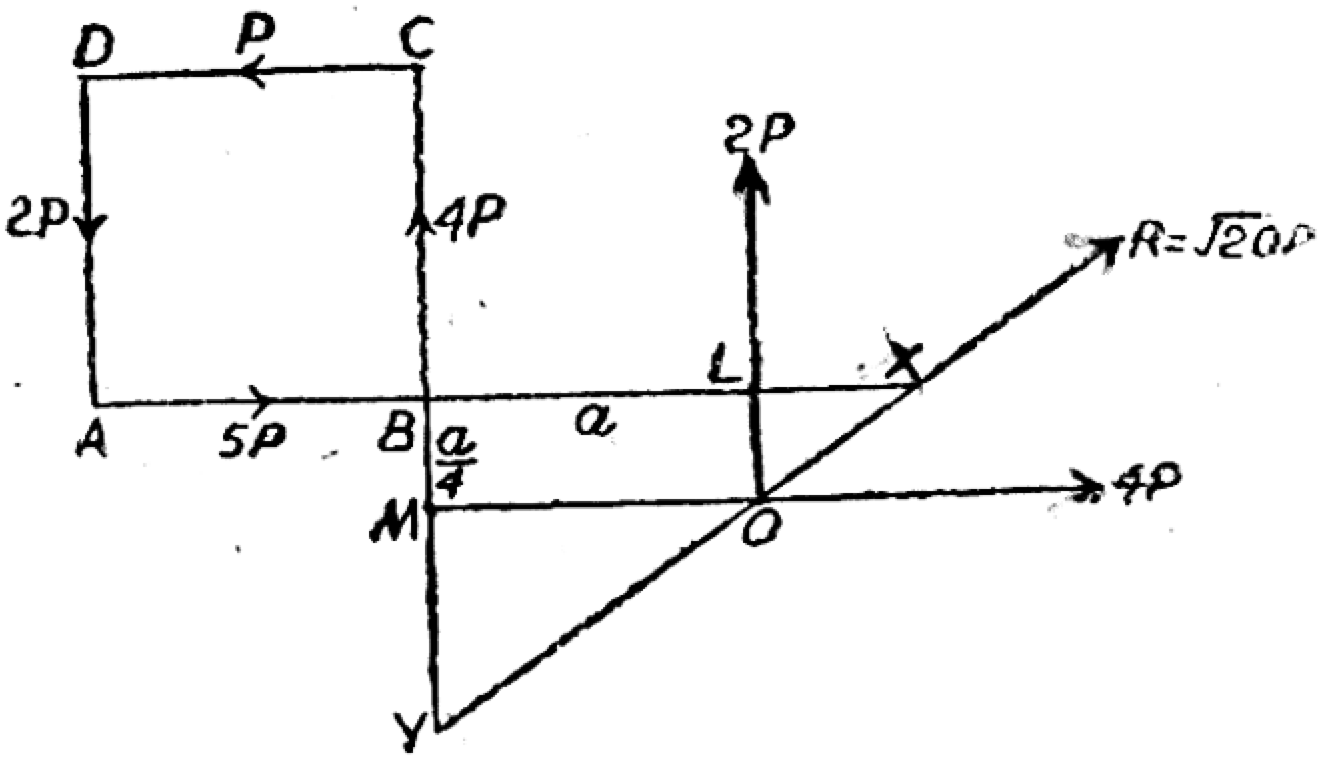
$$P.x - P.(x+a)\sin 45° = 0$$

$$\therefore \left(1 - \frac{1}{\sqrt{2}}\right)x = \frac{a}{\sqrt{2}}, \text{ or } x = \frac{a}{\sqrt{2}-1}.$$

If x were found negative, D would lie on the other side of C.

Note. The algebraic sum of the moments of the forces about A is obviously $-P.a$ = algebraic sum of the moments about B. It follows that the resultant is parallel to BA.

• **Ex. 3.** Forces 5P, 4P, P, 2P act along the sides AB, BC, CD, DA of a square, taken in order. Find the magnitude, direction and line of action of their resultant.



The resultant R of the forces must meet all the sides of the square (whose side is taken equal to a) since the forces along the parallel sides are unequal. Suppose R meets AB in X and BC in Y.

(*i*) The algebraic Sum of the moments about X (which is a point on the line of action of resultant) must be zero

$$\therefore\ 5P\times 0-4P\times BX+P\times a+2P(a+BX)=0$$

$$\therefore\ BX=\frac{3a}{2}.$$

(*ii*) Similarly, taking moments about Y, we have, $4P\times 0 + P(a+BY)+2P\times a-5P\times BY=0$

$$\therefore\ BY=\frac{3a}{4}.$$

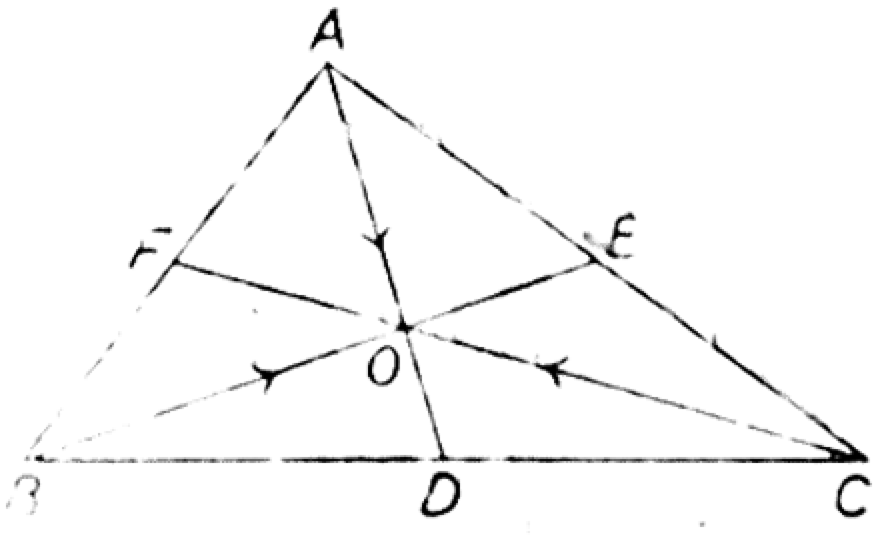
Again, the vertical forces (4P and 2P) will compound into a force 2P *upwards* acting at *L*, and the horizontal forces (5P and P) will compound into 4P acting at M. *The resolved parts of R* parallel to AB, BC are 4P and 2P meeting at O, which must be situate on the line of action of R.

$\therefore$ the final resultant $=\sqrt{(2P)^2+(4P)^2}=\sqrt{20}P=2\sqrt{5}P.$

Note. If BX had come out *negative*, we would have checked the Geometry by means of Algebra, and X would have been on the *left of* A in AB produced, at a distance $\frac{3a}{2}$ from B.

Similar remarks would apply to BY.

Ex. 4. ABC is a triangle ; D, E, F are the middle points of the sides BC, CA, AB respectively. Show that the forces represented by the straight lines AD, BE, CF, and acting at a point are in equilibrium.



Take moments about the vertex A.

Moment of force AD about A is zero

" " " BE " " is 2 area of $\triangle$ABE $=$ $+$area $\triangle$ABC

" " " CF " " is -2 area ACF $=-$area of $\triangle$ABC

$\therefore$ algebraic sum of the moments of the forces about A $=0+$area of $\triangle$ABC$-$area of $\triangle$ABC$=0$.

$\therefore$ The moment of the resultant R of the forces$=0$.

(*i*) $\therefore$ either $R=0$, or R must pass through A.

Similarly taking moments separately about B and C, we have

(*ii*) either $R=0$, or R must pass through both B and C.
But R cannot pass simultaneously through A, B and C.
Hence the Resultant R must be zero,
i.e., the forces are in equilibrium.

Simpler Exercises VII

1. Parallel forces 3, 5, 6, 4 act at equal distances of 8 inches along a rod; where must a prop be placed in order that the rod may rest in a horizontal position?

2. Five parallel forces, 1, 6, 3, 4, 8, act one foot apart on a straight horizontal rod; what force must be added to the force 1, in order that if the rod be supported where the force 3 acts, it may remain horizontal?

3. A rod is 6 feet long; parallel forces 6, 12, 18 act at the ends and middle point respectively, and at right angles to the rod: find the magnitude and point of application of the force which will produce equilibrium.

4. Four parallel forces, 1, 6, 9, 8 act at points 3 inches apart along a straight rod and at right angles to the rod; where must the rod be supported in order that it may remain in equilibrium?

5. A uniform rod has a vertical force W acting at its middle point and when suspended at a certain point, rests in a horizontal position with vertical forces W_0 and W_1 at its extremities or W_2 and W_0 at the same ends, respectively. What vertical force at one end will keep it horizontal?

6. AB is a diameter of a circle and BP and BQ are chords at right angles to each other; show that the moments of forces represented by BP and BQ about A are equal.

7. Masses of 2, 4, 8 and 16 lbs. are placed at a series of points in a line at right angles to the edge of a table and at distances of 4, 3, 2 and 1 ft. from the edge. Find the magnitude and point of application of the resultant force.

8. A uniform beam 20 ft. long and weighing 200 lbs. is supported on two props, one 6 ft. from one end and the other 9 ft. from the other end of the beam; calculate the pressure on each prop, when a man weighing 180 lbs. stands as near this latter end as he can without upsetting the beam. [London]

9. ABC is a right-angled triangle, the angle A being right and the sides *a*, *b*, *c*, being 5, 3, 4 units of length

respectively; the moments of a force F about A, B, C are 0, 16 and 9 units of moment, respectively. Find the magnitude and direction of F.

10. A uniform plank ABC of length 12 ft. and weight 80 lbs. rests on two supports A and B one at the end A and the other B, $4\frac{1}{2}$ ft. from end C. A boy walks along the plank from A to C and just as he reaches C, the plank commences to tilt. Find the weight of the boy. [M. U.]

11. Forces proportional to 1, 2, 3, 4 act along the sides AB, BC, AD and DC respectively, of the square ABCD, the length of whose side is 2 ft. Find the magnitude and line of action of their resultant.

12. Forces 3P, 4P, 5P act along the sides AB, BC and CA of an equiangular triangle ABC; find the magnitude, direction and position of their resultant.

13. Masses 2,3,4,5 are fastened to the rim of a wheel at points symmetrically arranged round the circumference. A string, coiled round the circumference, supports a weight of $\sqrt{2}$ lbs. If the wheel is free to rotate ina vertical plane about its centre ,prove that in the position of equilibrium, the radius to the first weight is inclined at 15° to the vertical.

14. At what point of a tree must one end of a rope of given length *a* be attached so that a man pulling at the other end with a given force may have the greatest tendency to pull it over? *(P. U. 1935)*

15. Forces 3, 7, 5 act along the sides AB, BC, CA of an equilateral △ ABC; find their resultant.

16. Forces 1, 2, 5, 4 act along the sides of a rectangle ABCD taken in order; find their resultant if AB=3 and BC=2.

17. ABCD is a square; along AB, CB AD and DC, equal forces P act; show that their resultant acts along DC and its magnitude is equal to double of any component.

18. A rod is supported by means of two strings which are attached to a fixed point and tied to each end of the rod. Assuming that the weight of the rod acts at its middle point, prove that the tensions of the strings are proportional to their lengths.

19. The moment of a force of given magnitude about two given points are known. Find geometrically its line of action; state all the possible solutions.

Ans. *The perpendicular distances p_1 and p_2 from the two given points A and B to the line of action of the force are given. Therefore, the line of action of the force is a common tangent to the two circles described with centres A and B and radii equal to p_1 and p_2, respectively.*

20. A uniform bar 4 feet long rests on a horizontal table with one end projecting 6 inches over the edge. If the weight of the bar be 10 lbs. find the maximum weight that can be suspended from this end so that the bar may not turn over.

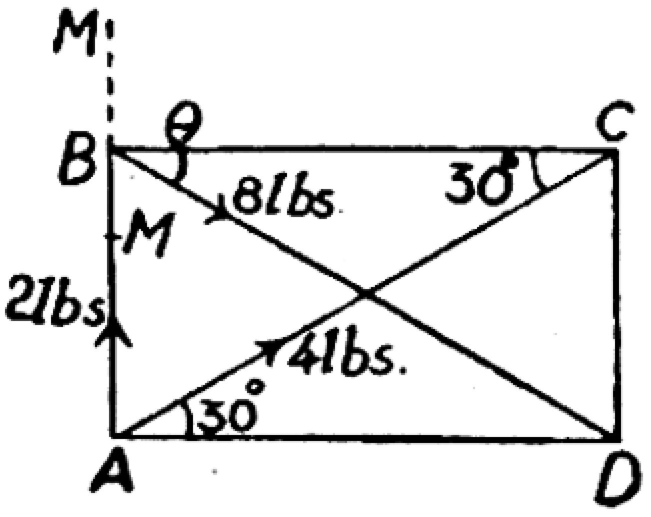
21. A light rod 3 ft. long rests on a horizontal position on two pegs which are at distances 15 and 9 inches from the two ends respectively. If each peg can support a pressure equal to the weight of 50 lbs. without giving way, find the greatest magnitude of equal weights that can be suspended from the two ends of the rod.

22. A uniform beam 12 ft. long and weighing 50 lbs. rests on two supports at equal distances from the ends.

Find the maximum value of this distance so that a man weighing 11 stone may stand anywhere on the beam without tilting it.

23. A uniform bar, 2 feet long and weighing 34 lbs. is suspended by two vertical strings. One is attached at a point 3 inches from one end and can just support a weight of 18 lbs. without breaking; the other is attached 4 inches from the other end and can just support 20 lbs. A weight of $3\frac{2}{5}$ lbs., is now attached to the rod; find the limits of the positions in which it can be attached without breaking either string [Delhi 1944.]

50. Example. *If ABCD is a rectangle such that the diagonal AC is 2 AB and forces of 2 lbs., 4 lbs., and 8 lbs. act along AB, AC, and BD, respectively, prove that their resultant is parallel to AD, and find its position.*



I Method. Since $\frac{AB}{AC} = \frac{1}{2}$,

$$\angle BCA = 30°.$$

Force of 4 lbs. has the components $2\sqrt{3}$ lbs. and 2 lbs. along AD and AB, respectively. Similarly 8 lbs. has the components $4\sqrt{3}$ lbs. and -4 lbs. along

BC and AB, respectively. The forces along AB vanish and we are left with forces $2\sqrt{3}$ along AD and $4\sqrt{3}$ along BC. These have obviously the resultant $6\sqrt{3}$ lbs. parallel to AD, acting at M in AB, such that AM/MB=2.

II Method. The general method of procedure will be as follows:—

Resolving the forces parallel to AD and AB, we have

$$X = 4 \cos 30^\circ + 8 \cos 30^\circ = 12 \frac{\sqrt{3}}{2} = 6\sqrt{3}.$$

$$Y = 2 + 4 \sin 30^\circ - 8 \sin 30^\circ = 0.$$

Hence the resultant has the magnitude $6\sqrt{3}$ lbs. and is parallel to AD. [*Cf.* Ex. 11, Art. 32.]

To find the line of action of the resultant, let us suppose that the resultant meets AB in M. Then taking moments of the forces about M, we have

$$4\ AM \sin 60^\circ + 8\ BM \sin 60^\circ = 0,$$

or $$AM + 2\ BM = 0,$$

or AM/BM = −2, showing that M is the *internal* point of division.

Exercises VII A

1. A uniform rod 6 ft. long and weighing 4 lbs. is supported by two vertical strings attached to its ends, and weights of 2, 3, 4, 5 lbs. are suspended from points on the rod 1, 2, 4, 5 ft. respectively from one end; find the tensions of the strings.

2. A uniform rod AB, 16 ft. long, weighs 2 lbs.; a wt. of 4 lbs. is hung from A, 1 lb. at a point 3 ft. from A, and a force of 11 lbs. wt. acts vertically upwards at a point 9 ft. from B; find where a suitable weight must be attached to the rod to keep it horizontal, and what is that weight?

3. A uniform beam AB, 10 ft. long, and weighing 20 lbs. is placed on two supports C and D equidistant from the middle point. A weight of 5 lbs. attached to A is on the point of disturbing the equilibrium; find what further weight should be attached to the point B so that it may just balance the beam about the support D. Also find the pressures on the supports.

4. A uniform beam 6 ft. long weighs 56 lbs. and is placed on two supports at its ends. If one of the supports can bear a maximum pressure of $1\frac{1}{2}$ cwt., find the least distance from this support at which a body of 3 cwt. can be suspended.

5. A heavy rod ACDB where $AC=a$ and $DB=b$, rests horizontally upon two smooth pegs C and D. If a load P were applied at A, it would just disturb the equilibrium, and similarly would do the load Q applied to B. If $CD=c$, prove that the weight of the rod $=(Pa+Qb)/c$. Determine where this weight acts.

6. Forces of 3 lbs., 2 lbs., 3 lbs., respectively, act along the sides AB, AD, CD of a square ABCD. Find the magnitude and line of action of the fourth force which will be in equilibrium with these three forces.

7. ABCD is a square and forces of 2 lbs., 4 lbs., 6 lbs., and 8 lbs., act along the sides AB, BC, CD, DA, respectively. Find the magnitude and line of action of the resultant.

8. If ABCD is a rectangle such that CD : AC :: 3 : 5 and forces of 3 lbs., 5 lbs., 5 lbs. act along AB, AC, DB, respectively, show that their resultant is parallel to AB; also find its magnitude and position.

9. Forces P,Q,R act on a rigid body along the sides of a right-angled isosceles triangle ABC (BC being the hypotenuse), taken the same way round; find the resultant.

10. Forces equal to the weights of 3. 5, 7 lbs. act along three of the sides of a square taken in order; find the magnitude of the force in the remaining side, which together with some force acting through the centre of the square will balance the three given forces. Find also the force through the centre. (*P. U. 1916.*)

11. A bicycle and its rider weigh 4 stone 10 lbs. and 8 stone 8 lbs., respectively, and their respective centres of gravity are at horizontal distances of 8 in. and 6 in. from the centre of the front wheel. Find the pressures on the ground exerted by the wheels, if their points of contact with the ground are 3 ft. apart.

12. Forces of 3 lbs. 5 lbs. and 7 lbs. wt. act along the sides AB, BC, CA of an equilateral triangle ABC. Find the

magnitude and direction of the resultant, and the point where it meets AB.

13. If three forces acting along the altitudes of a triangle are in equilibrium, show that they are proportional to the corresponding sides.

14. If O be the circumcentre of the triangle ABC and if forces act along OA, OB, OC, respectively proportional to the sides BC, CA, AB; show that their resultant passes through the incentre. (*P. U. 1937*)

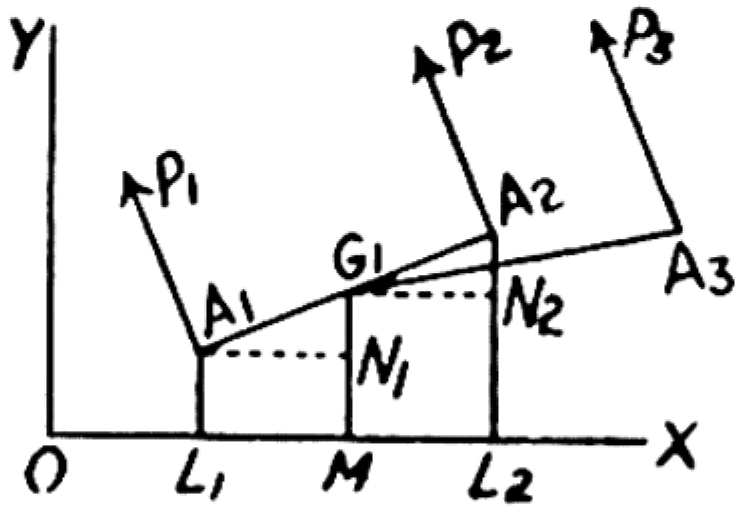
[*Hint*: If the bisector of ∠A meet BC in D, $(b+c)$. OD is the resultant of b. OB and c. OB.]

51. *To find the centre of a system of parallel forces.*

The centre of a system of parallel forces is the point at which the resultant of the system would act, whatever be the direction of the forces.

Let the like parallel forces Pr, act at points Ar, lying in the plane XOY and having co-ordinates $(\overline{x_r, y_r})$, $r = 1, 2, 3 \ldots n$, with respect to OX, OY as axes.

Let R_1 be the resultant of P_1 and P_2 acting at G_1, $(\overline{x_1, y_1})$.



Then $$\frac{P_1}{P_2} = \frac{G_1A_2}{A_1G_1} = \frac{G_1N_2}{A_1N_1} = \frac{x_2 - x_1}{\bar{x}_1 - x_1} \quad \ldots(1)$$

Also $$\frac{P_1}{P_2} = \frac{A_2N_2}{G_1N_1} = \frac{y_2 - \bar{y}_1}{y_1 - y_1} \quad \ldots(2)$$

From (1) $\bar{x}_1 = \dfrac{P_1x_1 + P_2x_2}{P_1 + P_2}$; from (2) $y_1 = \dfrac{P_1y_1 + P_2y_2}{P_1 + P_2}$.

Also $R_1 = P_1 + P_2$.

Applying this result to R_1 and P_3, we get

$$\bar{x}_2 = \frac{R_1\bar{x}_1 + P_3x_3}{R_1 + P_3} = \frac{P_1x_1 + P_2x_2 + P_3x_3}{P_1 + P_2 + P_3},$$

$$\bar{y}_2 = \frac{P_1y_1 + P_2y_2 + P_3y_3}{P_1 + P_2 + P_3},$$

and $R_2 = P_1 + P_2 + P_3$.

By this process the result can be extended to any number of like parallel forces.

Hence if $(\bar{x},\ \bar{y})$ be the point of application of resultant R of the forces, $R=\Sigma P_r$,

$$\bar{x}=\frac{\Sigma P_r x_r}{\Sigma P_r},\ \bar{y}=\frac{\Sigma P_r y_r}{\Sigma P_r}.$$

Note 1. If A_1, A_2 are kept fixed and P_1, P_2 are kept of the same magnitude, but are changed in direction, remaining parallel to each other, the point G_1 will not change, since there is only one point which divides a given line in a given ratio.

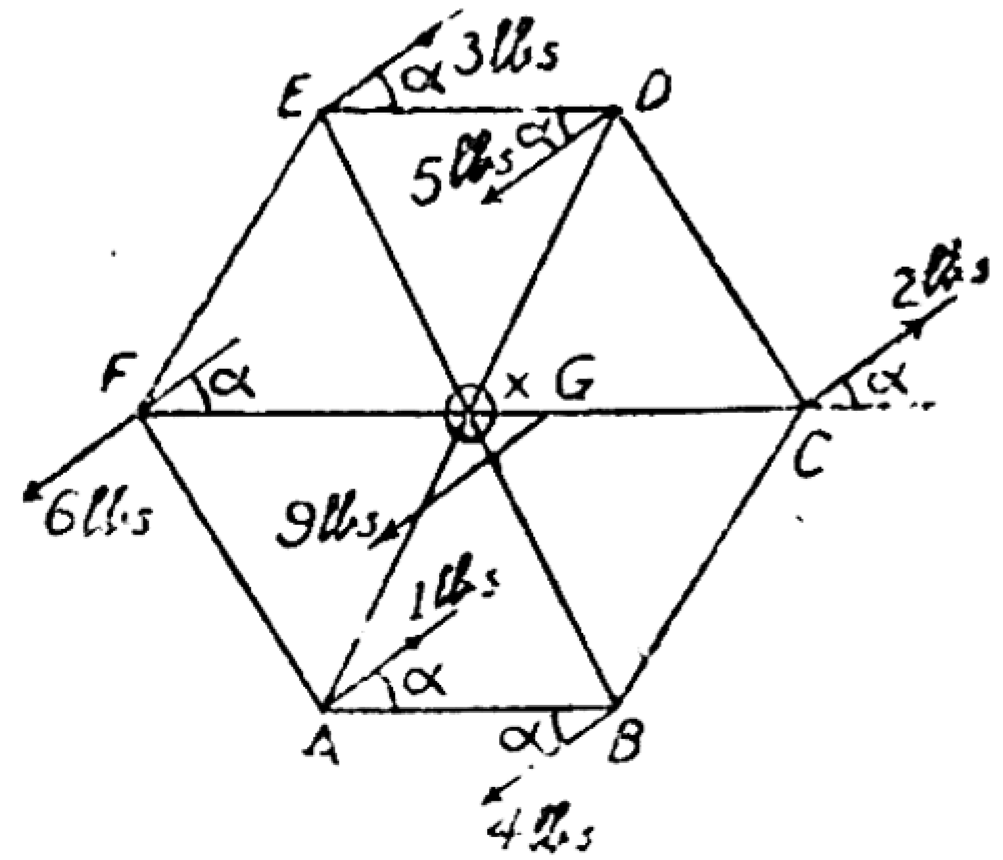
[See also Note 3, Art. 40.]

The resultant R_1 would also turn about G_1 and become parallel to the changed direction of the forces. It follows that the centre of parallel forces is independent of the direction of the forces.

Note 2. The formulae will be easily seen to hold even when some of the forces have the opposite direction, provided such forces are given negative signs.

52. Ex. 1. *Parallel forces of 1, 2, 3 lbs. wt. act at the corners A, C, E of a regular hexagon ABCDEF, and forces, 4, 5, 6, lbs. wt. act at the corners B, D, F in a direction parallel to the first three but in the opposite sense. Find the point of application of the resultant.* [*P. U. 1926.*]

Let O be the centre of the hexagon. Then AD, BE, CF are all concurrent at O.



Let the resultant meet FC at G at a distance x from O. Let the sides of the hexagon be r.

It would be convenient to take moments of all the forces about O, since O is symmetrically situated with respect to the corners of the hexagon, where the forces act.

Let the forces make any angle α with AB, FC, or ED, which are all parallel.

Now since the algebraic sum of the moments of the component forces about O is equal to the moment of the resultant, 9 lbs. wt. (acting in the opposite sense), about O, we have

1. $r \sin (60^\circ - \alpha) - 4r \sin (120^\circ - \alpha) + 2r \sin \alpha$
$+ 5r \sin (60^\circ - \alpha) - 3r \sin (60^\circ + \alpha) + 6r \sin \alpha = -9x \sin$,
or $r \{ 6 \sin (60^\circ - \alpha) - 7 \sin (60^\circ + \alpha) + 8 \sin \alpha \}$
$= -9x \sin \alpha$, [$\because \sin (120^\circ - \alpha) = \sin (60^\circ + \alpha)$]

or
$$r \left\{ 6 \left(\frac{\sqrt{3}}{2} \cos \alpha - \frac{1}{2} \sin \alpha \right) \right.$$

$$\left. -7 \left(\frac{\sqrt{3}}{2} \cos \alpha + \frac{1}{2} \sin \alpha \right) + 8 \sin \alpha \right\} = -9x \sin \alpha,$$

or, $r \left\{ -\frac{\sqrt{3}}{2} \cos \alpha + \frac{3}{2} \sin \alpha \right\} = -9x \sin \alpha,$

or, $+r\sqrt{3} \cos (60^\circ + \alpha) = 9x \sin \alpha.$

Hence $x = \dfrac{r\sqrt{3} \cos (60^\circ + \alpha)}{9 \sin \alpha}$. ...(1)

This point where the resultant meets FC, depends upon α.

The result can be verified by considering the special cases, when $\alpha = 0^\circ$, *30°, or 90°.*

When $\alpha = 0$, the forces become all parallel to FC and therefore, their resultant is also parallel to FC, and meets it at infinity, which is the value of x given by the result (1).

Otherwise. We may find the *centre* of the parallel forces as follows :—

Choosing OC as x-axis and a st. line perp. to OC through O as y-axis, $(\bar{x}, \bar{y})$, the co-ordinates of the centre, will be found even though all the forces are not *like*, by giving the negative sign to the *unlike* forces.

The co-ordinates of A are $\left(-\frac{r}{2}, -\frac{r\sqrt{3}}{2} \right)$; of B, $\left(\frac{r}{2}, -\frac{r\sqrt{3}}{2} \right)$; of C, $(r, 0)$; of D, $\left(\frac{r}{2}, \frac{r\sqrt{3}}{2} \right)$; of E, $\left(-\frac{r}{2}, \frac{r\sqrt{3}}{2} \right)$; and of F, $(-r, 0)$.

$$\therefore \bar{x} = \frac{-\frac{r}{2} - 4 \cdot \frac{r}{2} + 2.r - 5 \frac{r}{2} \cdot + 8 \cdot \left(-\frac{r}{2} \right) - 6(-r)}{1 - 4 + 2 - 5 + 3 - 6}$$

$$= \frac{-r}{6},$$

$$\bar{y} = \frac{\frac{-r\sqrt{3}}{2} + 4 \cdot \frac{r\sqrt{3}}{2} - 5 \cdot \frac{r\sqrt{3}}{2} + 3 \cdot \frac{r\sqrt{3}}{2}}{1-4+2-5+3-6} = \frac{-r\sqrt{3}}{18}.$$

Thus the *centre* of the parallel forces is $\left(-\frac{r}{6}, -\frac{r\sqrt{3}}{18}\right)$.

This point is independent of α, which means that the *centre* is always the same, whatever the direction of the parallel forces may be.

The equation of the line of action of the force, which depends upon α is given by

$$y + \frac{r\sqrt{3}}{18} = \left(x + \frac{r}{6}\right) \tan \alpha,$$

Where it cuts x-axis, $y=0$.

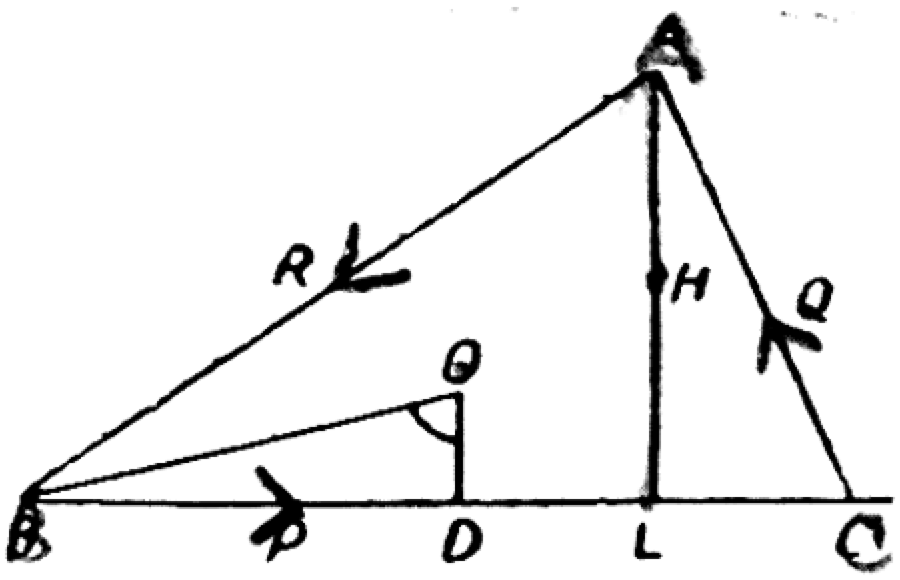
$$\therefore\ x = \frac{r\sqrt{3}}{18} \cot \alpha - \frac{r}{6}$$

$$= \frac{r\sqrt{3}}{9 \sin \alpha}\left(\frac{1}{2} \cos \alpha - \frac{\sqrt{3}}{2} \sin \alpha\right) = OG. \quad \{\text{from 1)}$$

Ex. 2. *Forces act along the three sides of a triangle ABC such that their resultant passes through the orthocentre and the circumcentre. Prove that the forces are in the ratio :—*

$$\frac{\cos B}{\cos C} - \frac{\cos C}{\cos B} : \frac{\cos C}{\cos A} - \frac{\cos A}{\cos C} : \frac{\cos A}{\cos B} - \frac{\cos B}{\cos A}.$$

(*P. U. 1922.*)



Let the forces P, Q, R act along the sides BC, CA, AB.

Let O, H *be the circumcentre and orthocentre*, OD and AL perpendiculars from O and A, respectively on BC.

Then OD=OB cos A= R′ cos A, where R′ *is the circumradius* ; and similarly for distances of other forces from O.

Also HL=LC tan LCH=LC cot B=b cos C cot B

$$= b \cos C \cos B/\sin B$$

$$=2R' \cos B \cos C \left[\because \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}\right.$$
$$\left. =2R'. \right]$$

It should be noted that the distance H L has been put in the symmetrical form, so that the result may apply to other forces in a similar way.

Now since the resultant acts along OH, its moment and hence the algebraic sum of the moments of the forces about each of the points O and H vanishes.

Hence taking moments about O,

$$P \cos A+Q \cos B+R \cos C=O \qquad \ldots(1).$$

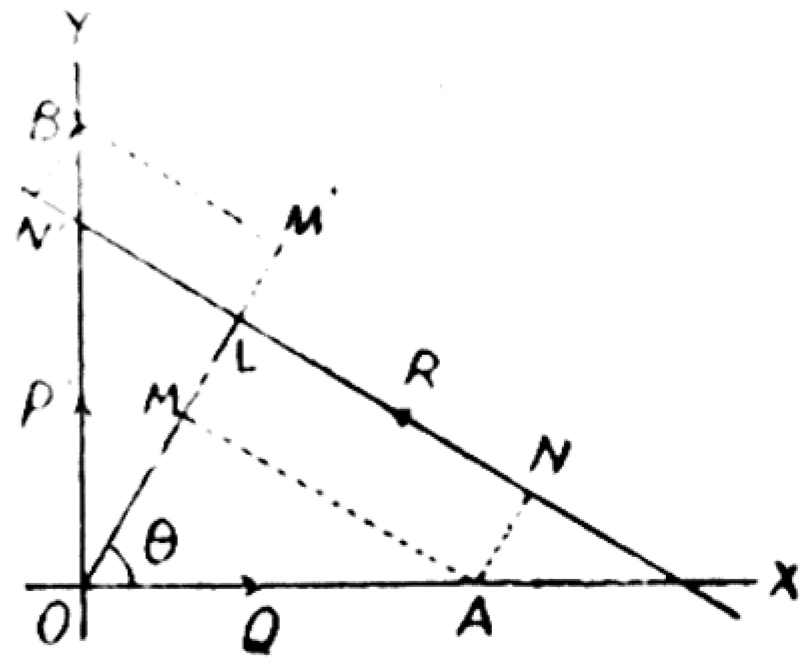
Taking moments about H,

$$P \cos B \cos C+Q \cos C \cos A+R \cos A \cos B=O$$

or
$$\frac{P}{\cos A}+\frac{Q}{\cos B}+\frac{R}{\cos C}=O \qquad \ldots(2).$$

From (1) and (2), by cross-multiplication, the ratios of P, Q, R are easily found.

***Ex. 3.** *Forces P, Q, R along the lines $x=0$, $y=0$ and $x \cos \theta+y \sin \theta=p$. Find the magnitude of the resultant and the equation of its line of action. (The axes of co-ordinates are rectangular).* [*P. U. 1926.*]



OX, OY being the axes, let NN′ represent the line $x \cos \theta+y \sin \theta=p$, so that $OL=p$ and $\angle XOL=\theta$.

(*i*) *To find the magnitude of the resultant.*

Resolving the forces along the axes.

$$X=Q-R \sin \theta,$$
$$Y=P+R \cos \theta.$$

Hence the resultant $=\sqrt{(Q-R \sin \theta)^2+(P+R \cos \theta)^2}$
$=\sqrt{P^2+Q^2+R^2+2R(P \cos \theta-Q \sin \theta)}$.

(*ii*) Now *to find the equation of the line of action*, let us suppose that it meets the axes in A, B respectively, where A is $(x_1, 0)$ and B is $(0, y_1)$.

Taking moments of the forces about A and B,

$$-Px_1+R.\ AN=0, \text{ and } Qy_1-R.\ BN'=0.$$

But $AN=ML=OL-OM=p-x_1 \cos \theta$

and $BN' = OM' - OL = y_1 \sin\theta - p$.

Hence $-Px_1 + R(p - x_1\cos\theta) = 0$,

and $Qy_1 - R(y_1\sin\theta - p) = 0$,

i.e., $x_1 = Rp/(P + R\cos\theta)$ and $y_1 = Rp/(R\sin\theta - Q)$.

Hence the equation of AB is $x/x_1 + y/y_1 = 1$

or $x(P + R\cos\theta) + y(R\sin\theta - Q) = Rp$

or $xP - yQ + R(x\cos\theta + y\sin\theta - p) = 0$.

Exercises VII B

1. Explain what is meant by the centre of parallel forces.

If like parallel forces of equal magnitude act at the vertices of a triangle, show that the centre of the parallel forces coincides with the centroid of the triangle.

2. If equal and like parallel forces act at the vertices of a quadrilateral, find the centre of the forces.

3. Particles of weights 1, 2, 3, 4, 5, 6 lbs., respectively, are arranged in this order round the circumference of a circle of radius 10 inches at intervals of 60°. Find the distance of their centroid from the centre of the circle. [*P. U. 1936*]

Hint. Take the centre of the circle as the origin, and find $(\bar{x})^2 + (\bar{y})^2$.

4. Forces P, Q, R act along the sides BC, CA, AB, respectively, of a triangle ABC. If the resultant passes through the orthocentre, show that $P/\cos A + Q/\cos B + R/\cos C = 0$.

5. Forces P, Q, R act from the angular points A, B, C of a triangle ABC, perpendicular to the opposite sides. Prove that if their resultant passes through the centre of the circumscribing circle,

$P(b\cos C - c\cos B) + Q(c\cos A - a\cos C) + R(a\cos B - b\cos A) = 0$. (*P. U. 1924*)

6. Forces P, Q, R act along the sides BC, CA, AB of a triangle ABC, show that if their resultant passes through the orthocentre and centre of gravity of the triangle,

$P : Q : R :: \sin 2A \sin(B - C) : \sin 2B \sin(C - A) : \sin 2C \sin(A - B)$.

7. If three forces act as in Ex. 6 above, and their resultant passes through the incentre and circumcentre of the triangle, show that the forces are in the ratio

$\cos B - \cos C : \cos C - \cos A : \cos A - \cos B$.

8. Three forces act along the bisectors of the angles A, B, C of a triangle ABC. Show that if the forces be in equilibrium, they are in the ratio

$$\cos\frac{A}{2} : \cos\frac{B}{2} : \cos\frac{C}{2}. \quad (P.\ U.\ 1937)$$

9. Along the sides of a regular hexagon, taken in order, act six forces represented in magnitude by 1, 2, 3, 4, 5, 6 respectively. Prove that their resultant will be represented by 6, that its direction will be parallel to one of the sides and that its line of action will be at a distance from the centre of the hexagon equal to $3\frac{1}{2}$ times the radius of the inscribed circle. *(P. U. 1917)*

10. ABCDEF is a regular hexagon. Equal forces P act along AC, BD, CE, DF, EA, BF; show that their resultant is 2P parallel to BF and find its line of action. *(P. U. 1918)*

Hint.– Take moments about the centre of the hexagon.

11. Three like parallel forces P, Q, R act at the corners of a triangle ABC; prove that their resultant passes through the circumcentre of the triangle for all directions of the forces if P, Q, R are in the ratios sin 2A : sin 2B : sin 2C. *(P. U. 1939)*

Hint: If O is the circumcentre and if AO produced meets BC in D, prove that

BD : DC as sin BOD : sin DOC, or as sin 2C : sin 2B.

12. A rod AB of weight 10 lbs. is supported by a hinge at A, and carries a weight of 30 lbs. at B. It is tied back by a string attached at a point C in the rod, to a point D vertically above A. Both the rod and the string make angles of 45° with the vertical. If AB = 3AC, find the tension of the string. *(P. U. 1931)*

Hint. Take moments of the forces about A.

13. Four forces acting along the sides of a quadrilateral are in equilibrium; prove that the quadrilateral is plane. If also the quadrilateral can be inscribed in a circle, prove that each force is proportional to the opposite side. *(P. U. 1927)*

Hint. Take moments about the vertices.

14. Four coplanar forces act on a rigid body, their lines of action being the sides of a square ABCD. Their magnitudes and directions are (1) 20 lbs. along AB, (2) −10 lbs. along BC, (3) 30 lbs. along CD, (4) −40 lbs. along DA. Find the forces

P and Q whose lines of action are respectively CA and a line through the point B, which will produce equilibrium. Give also the direction of Q. *(P. U. 1931)*

Revision Paper I

1. A weight of 24 lbs. is suspended by two flexible strings one of which is horizontal, and the other is inclined at an angle of 30° to the vertical. What is the tension in each string? *(P. U. 1935 Supp.)*

2. Two forces given in magnitude act each through a fixed point, and are inclined at a constant angle θ; show that their resultant also passes through a fixed point A. If θ varies, show that the locus of A is a circle. *(P. U. 1937 Supp.)*

3. A uniform cubical block of weight W rests on a smooth inclined plane with one of its edges horizontal and in contact with a smooth vertical wall. The inclination of the plane is θ. Find the pressures on the wall and on the plane. *(P. U. 1940)*

4. Show that the resultant of two forces F_1 and F_2 inclined at an angle α to each other is $\sqrt{F_1^2+F_2^2+2F_1F_2\cos\alpha}$, and that the resultant makes with F_1 an angle

$$\sin^{-1}\left(\frac{F_2\sin\alpha}{\sqrt{F_1^2+F_2^2+2F_1F_2\cos\alpha}}\right).$$

If in the above case $F_2 > F_1$ in magnitude, prove that if the resultant makes a right angle with the direction of F_1, then

$$\alpha=\frac{\pi}{2}+\sin^{-1}\frac{F_1}{F_2}.$$ *(P. U. 1940 Supp.)*

5. ABCDEF is a regular hexagon, and at A forces act represented in magnitude and direction by AB, 2AC, 3AD, 4AE, 5AF; show that the length of the line representing their resultant is $\sqrt{351}$AB. *(P. U. 1933 Supp.)*

6. A triangle ABC can turn freely in its own plane about the centre of the inscribed circle which is fixed, and forces proportional to $y-z$, $z-x$, $x-y$ act along the sides BC, CA, and AB respectively. Show that the triangle remains at rest. *(P. U. 1937 Supp.)*

7. A string of length l is fastened to two points A, B at the same level at a distance a apart. A ring of weight W can slide on string, and a horizontal force x is applied to it such that it is in equilibrium vertically beneath B. Prove that $x=\frac{a}{l}$ W and that the tension of the string is $\frac{W(l^2+a^2.)}{2l^2}$

(*P. U. 1941*)

8. ABC is an isosceles triangle whose angle A = 120°, and forces of magnitude 1, 1, $\sqrt{3}$ lbs. weight act along AB, AC and BC; show that the resultant bisects BC and is parallel to one of the other sides of the triangle. (*P. U. 1941*)

9. ABCD is a square. Along AB, BC, CD, DA act forces F, 2F, 3F, 4F respectively. Show that their resultant is parallel to CA and meets CD produced, at P such that DP $=\frac{3}{2}$ DC. (*P. U. 1940 Supp.*)

10. Forces P, 3P, 2P, 5P act along the sides AB, BC, CD, DA of a square ABCD. Find the magnitude and direction of their resultant and prove that it meets AD produced at a point E such that AE : DE = 5 : 4. (*P. U. 1940*)

11. A light table stands on three equal vertical legs and a weight is placed at the centre of the circle inscribed in the triangle formed by the points of intersection of the legs with the table. Show that the pressures on the legs are proportional to the lengths of the opposite sides of the triangle.

(*P. U. 1941*)

12. A rod, of length $2l$ and weight W, is lying across two pegs on the same level a feet apart. If neither peg can bear a pressure greater than P, find the greatest length of the rod which may project beyond either peg. (*D. U. 1937*)

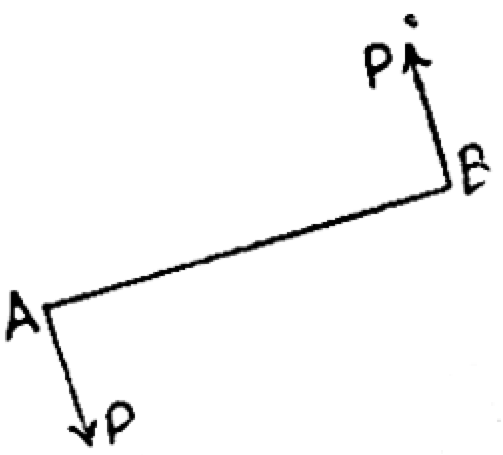
13. A beam 20 feet long rests on two supports 16 feet apart and overhangs the left hand support by 3 feet and the right hand support by 1 foot. It carries a load of 5 tons at the left hand end of the beam and one of 7 tons midway between the supports. The weight of the beam which may be looked upon as a load at its centre is 1 ton. Find the reactions at the supports. What upward force at the right hand end of the beam would be necessary to tilt the beam? (*P. U. 1945*)

CHAPTER V

COUPLES

53. Definition. *Two* **equal unlike parallel** *forces not having the same lines of action* and *acting upon a rigid body* constitute a **Couple.**

We have already seen (Art 40, Note 4) that *two equal and opposite forces, not having the same line of action cannot have a single resultant.* A pair of such forces is called a Couple. From Newton's second Law of Motion, it is apparent that when a single force is the resultant of any number of forces, the body moves in the direction of the resultant, *i.e.*, the body has what we called the *motion of translation.* Consequently, the forces of a Couple cannot have the effect of giving the motion of translation to the body.



The perpendicular distance AB between the lines of action of the constituent forces P, P of the couple is called the **arm of the Couple** and may be denoted by p.

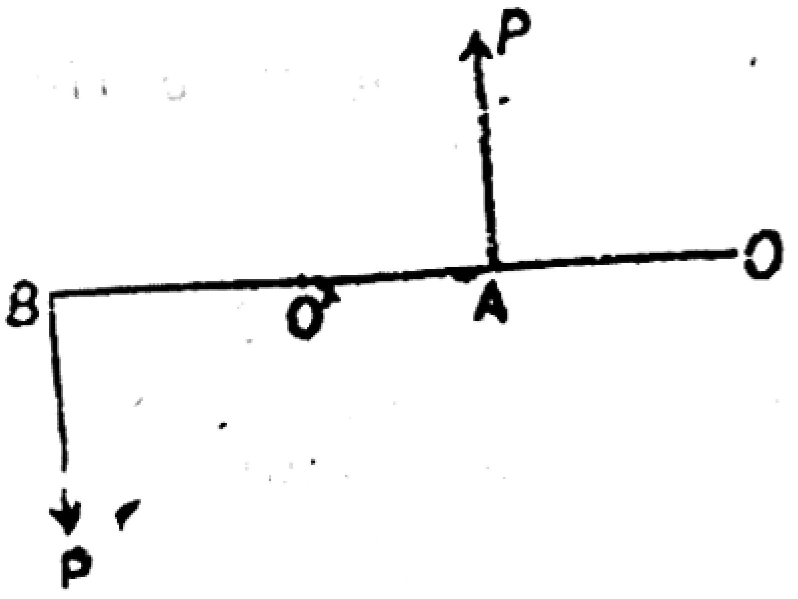
The product P.p is called the **Moment of the Couple** where P is the magnitude of either force of the Couple and p is the arm of the couple.

This couple will be symbolically represented by (P. p).

The student can easily verify that the *moment of a couple is also equal to the moment of either of its constituent forces about any point in the line of action of the other.*

54. *The [illegible]ebraic sum of the moments of the two forces of a couple abo[illegible] any point in their plane is constant and is equal to the* **moment of the Couple.**

Let (P, p) be a couple and let O be any point in the plane of the forces. Through O draw OAB perpendicular to the forces meeting them in A and B.



The algebraic sum of the moments of the forces P, P about O

$= P.\ OB - P.\ OA = P,\ AB = Pp.$

$=$ moment of the Couple.

Again, if we take moments about

a point O′ situated between the forces, the algebraic sum of moments about O′.

$$=P.\ O'B+P.\ O'A=P\ (O'B+O'A)$$
$$=P.\ AB=\text{moment of the Couple.}$$

Hence the proposition is true wherever the point O may be taken in the plane of the forces.

Note.—The moment of a Couple measures the *turning effect* of a couple on the body on which it acts and is called a *torque.*

Sign of a Couple. We have already seen that a couple acting on a body has a rotatory effect on it measured by the moment of the couple, the couple is said to be positive when it produces rotation in the *contra-clockwise direction, and negative* when the rotation produced is in the *clockwise direction.*

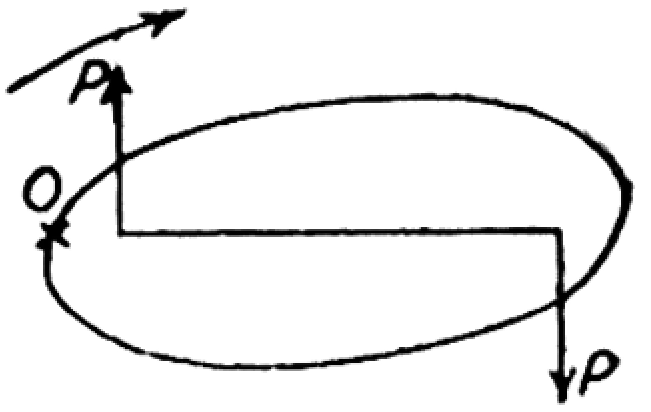


Fig. 1.

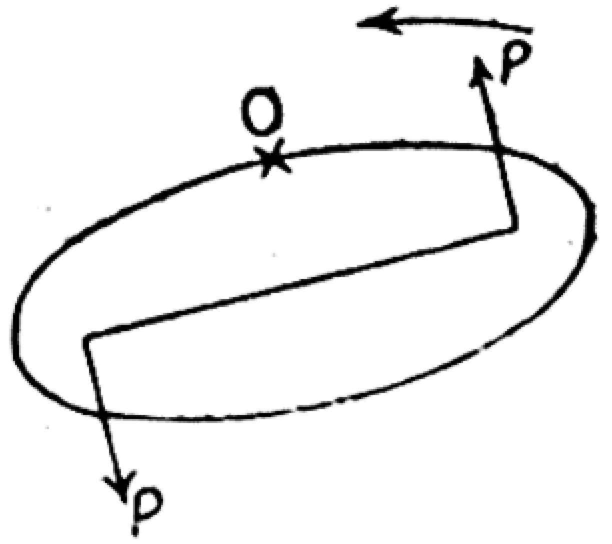


Fig. 2.

Thus, the Couple is negative in Fig. 1, and positive in Fig. 2.

Axis of the Couple. The axis of a Couple is a line perpendicular to the plane of the couple through *any point* in this plane and of such length as to indicate the magnitude of its moment.

The direction of rotation produced by a couple (*sense of the couple*) is indicated by the sign of its axis which is determined as follows:

"Place a watch on the plane of the Couple face upwards; if the direction of rotation is contra-clockwise, the axis is drawn *upwards*, and is considered *positive*; if clockwise, the axis is drawn *downwards*, and is considered negative."

In the figures (1) and (2), if the bodies be free to move, and be pivoted at *any points* O,O′ of the bodies in the plane of the couples, they would rotate in the directions indicated. This can be experimentally verified.

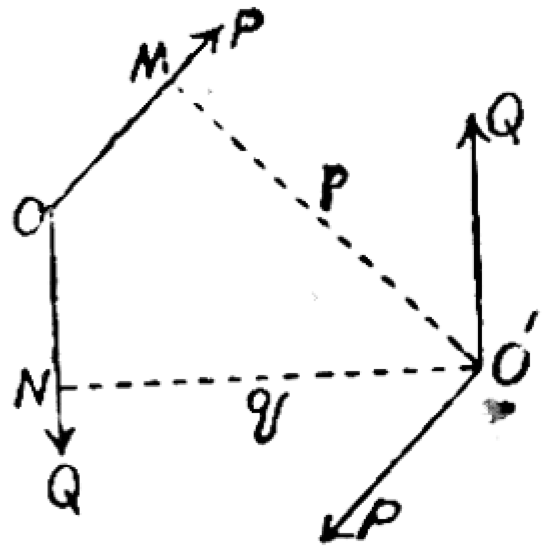
Note.—The student should carefully note that every rotation takes place about an axis and when we say that the moment of a force about a point O measures its turning effect, the turning of the body really takes place about an axis passing through O and perpendicular to the plane through the lines of action of the forces and the point O.

55. *Two couples acting in the same plane upon a rigid body whose moments are equal in magnitude but opposite in sign, balance each other.*

Let the two couples (P, p) and (Q, q) be of equal and opposite moments, *i.e.* $P.p=Q.q$.

Case I. *When the forces P, and Q are not parallel.*

Let one of the forces P meet one of the forces Q at O and let the other two forces meet at O′.



Draw O′M, O′N perpendiculars to the directions of P and Q passing through O; O′M=*arm p* and O′N=*arm q*.

Now, the sum of the moments about O′ of P and Q at O

$=P.\ O'M-Q,\ O'N=Pp-Qq,$

$=0\ [\because Pp-Q.q=0,$ by hypothesis]

$\therefore$ The resultant R of P and Q acting at O passes through O′.

i e., R must act *along* O O′.

Similarly, the resultant R′ of P and Q acting at O′ must act *along* O′O so that R′ acts opposite to the direction of R′.

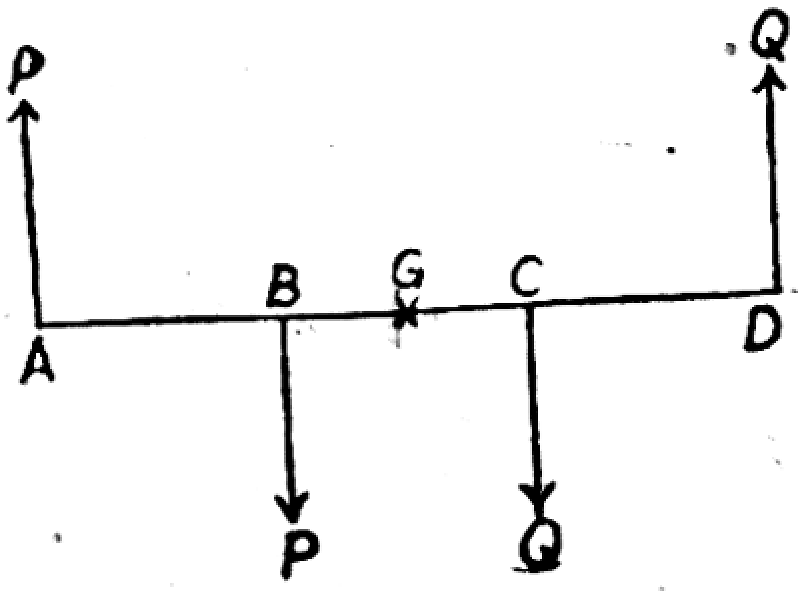
Again $R=\sqrt{P^2+Q^2+2\,PQ\cos\alpha}=R'$

Where α is the measure of the angle at O and O′ between P, Q at O and their parallels P′, Q′ at O′.

Hence, R at O and R′ at O′ being equal and opposite balance each other, *i.e.*, the four forces of the P-couple and the Q-couple balance each other.

Case II. Let the four forces of the two couples, be all parallel.

Let any straight line perp. to their directions cut their lines of action in A, B, C, D.



We are given that,

$P.\ AB=Q.CD$ ... (1)

The resultant of P at A and Q at D is P+Q, acting at a point G, which is such that $P.\ AG=Q.\ GD$ ... (2)

Subtracting (1) from (2),

$P.\ BG=Q.\ CG$ ... (3)

But the resultant of the two forces at B and C is also $P+Q$, but in the opposite direction; and (3) shows that G is the point where it acts.

Hence all the four forces of the **P** couple and the **Q** couple are in equilibrium.

56. Since two co-planar couples of equal moments but of opposite signs, balance one another, we get, by *reversing the forces* of one of the balancing couples, the following propositions.

(*i*) *Any two co-planar couples of equal moments are equivalent.*

(*ii*) *Two like couples of equal moment are equivalent to a couple of double the moment.*

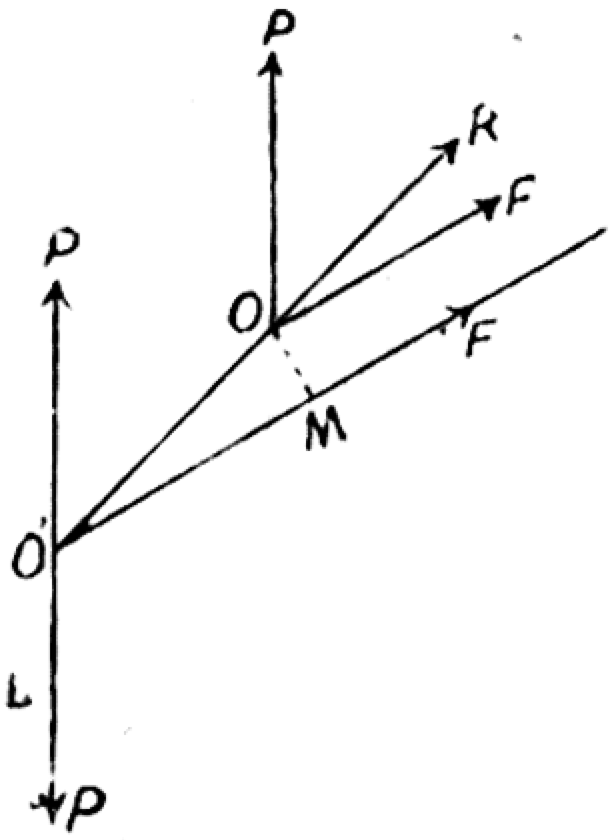
Resultant of a single force and a couple.

57. *A couple and a force in the same plane cannot balance but are equivalent to a single force equal and parallel to the given force and having the same direction.*

Let (P, p) be the couple, and let F be the force.

To prove that the resultant of P, P, F is a single force which is equal in magnitude and has the same direction as the given force F.

Case I. *P and F are not parallel.*



Let F intersect P (one of the forces of the couple) at O, and let the resultant of P and F at O be R. Produce R to meet the other force **P** at O′.

At O, the components of **R** are **P** and F.

When the point of application of **R** is transferred to O′, the components of **R** at O′ must be **P** and F, (the same as at O) in directions parallel to the respective directions of the components of **R** at O.

We have now two equal and opposite forces P and P at O′ and a force F at O′ parallel to its original direction.

Now P and P at O′ balance. Hence **F acting at O′** parallel to its original direction *is the resultant of the system.*

To find the distance OM between the resultant force F at O′ and the given force F at O.

Now F at O′=resultant of P, F at O and P at O′ acting along O′L

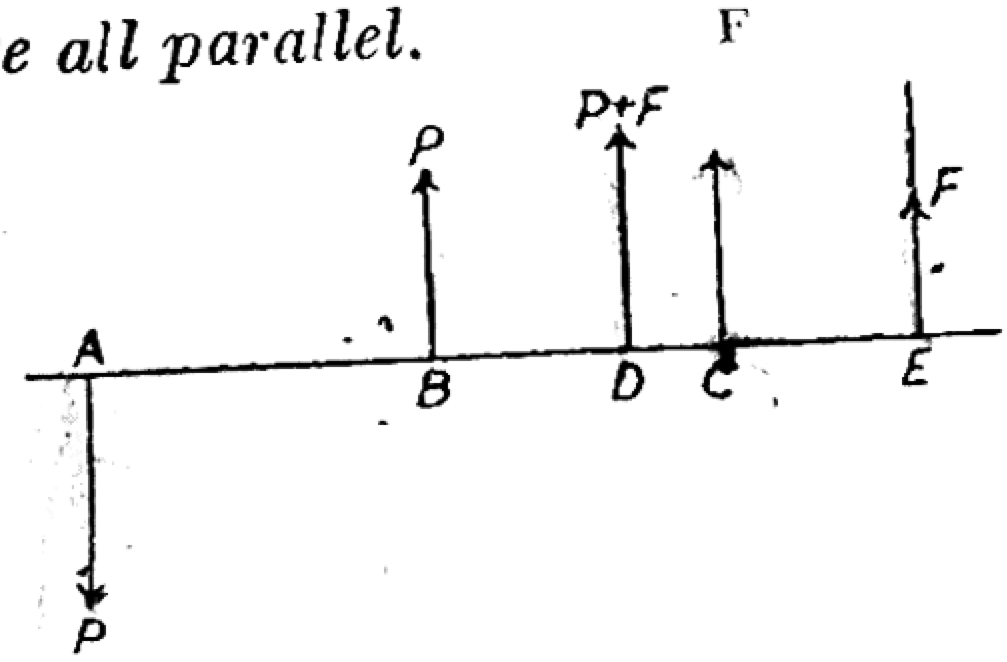
∴ Taking moments about O, we have

F×OM=moment of F about O (which is zero)

+moment of the couple (which is Pp) =P.p

$$\therefore \text{OM} = \frac{\text{Pp}}{\text{F}}.$$

Case II. *Forces P and F are all parallel.*



Draw a line perpendicular to the direction of the forces to meet them in the points A, B, C as shown.

Now, the resultant of P at B and F at C is a force P+F parallel to them and acting at a point D such that DC×F=BD×P.

Again, combining the unlike parallel forces P at A and P+F at D, we get as *the final resultant* a force F, parallel to them and acting at a point E in AD produced, such that ED× (P+F)=EA×P.

Hence F acting at E parallel to its original direction is the resultant of the system.

Now F at E is the resultant of F at C, P at B and P at A acting in the opposite direction.

Taking moments about C, we have

F×CE=moment of F about C (which is zero)

+moment of the couple Pp.

$$\therefore \text{CE} = \frac{\text{P.p}}{\text{F}}$$

which gives the distance between the final resultant F at E and the given force F at C.

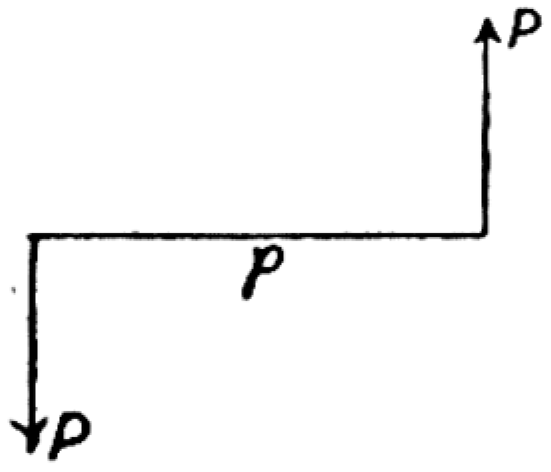
58. Alternative Proof of the theorem in Art. 57 Resultant of a single force and a couple.

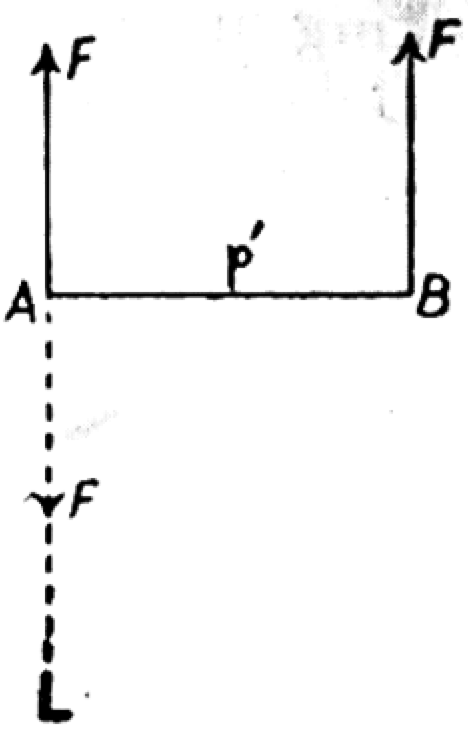
Let the given couple be (P, p), and let the given force F be acting at A.

We can replace the couple by another couple of equal moment Art. 56.

Let it be replaced by the equivalent couple (F, p′) so that P.p=F.p′, *i.e.* $p' = \frac{\text{P.p}}{\text{F}}$.

The new couple is to be taken in such a way that one of its forces F acts along AL, opposite to the direction of the given force F.





The given force F at A and F along AL, balance and we are left with the single force F acting at B, where AB = p'

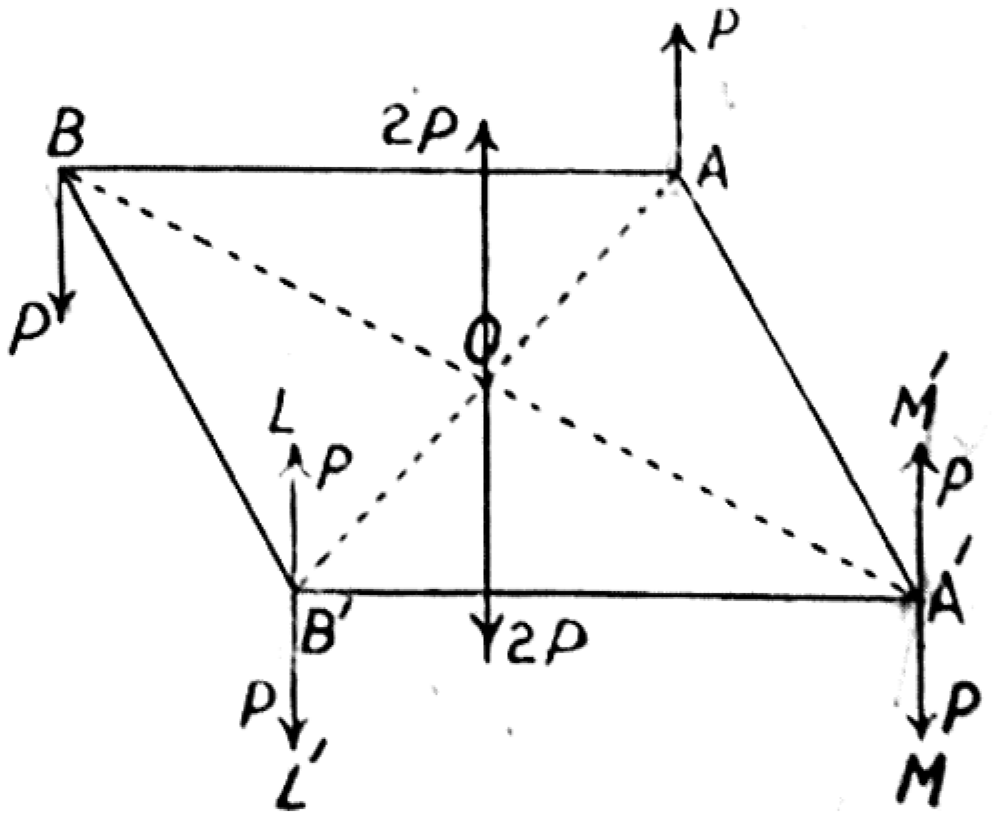
$$= \frac{P.p}{F}.$$

Hence, F acting at B parallel to its original direction is the resultant of the system. *Conversely, a force may be transferred to any arbitrarily chosen point by the introduction of a couple.*

Note 1.—This proof is *very general*, and holds equally well whether the force F be parallel, or not, to the forces of the couple.

Note 2.—The given couple, as shown in the figure has a positive moment; so the couple (F, p') replacing it, must have the same sign for its moment. If (P,p) had a negative moment, the arm A should have been drawn in the opposite direction. So whether the resultant force is to the right or left of the point A, would depend upon the sign of the moment of the given couple.

59. Theorem.—*The effect of a couple upon a rigid body is not altered by moving the couple, so that the arm moves parallel to itself, in its own plane or in a parallel plane.*



Let (P, AB) be any given couple. Let the couple be moved so that the arm occupies the position A'B', in its own plane or in a parallel plane. Introduce at A' and also at B', two opposite forces, each equal to P in magnitude, and parallel to the original forces of the couple. These forces do not affect the system in any way. Since any two parallel lines are coplanar, AB and A'B' lie in the same plane, and hence AB'

A'B intersect at O (say). This point O is the middle point of AB' and also of A'B, since AA'B'B is a parallelogram.

Now the resultant of P at A and P at B' along B'L is 2P at O. Similarly, the resultant of $-P$ at B and $-P$ at A' (along A'M) is $-2P$ at O. These two resultants destroy each other and we are left with the forces, P at A' along A'M' and $-P$ at B' (along B'L') which constitute the couple (P, A' B'). If A'B' is in a parallel plane, this couple will lie in that plane.

60. Composition of couples.—*The resultant of any number of couples in a plane is a couple whose moment is the algebraic sum of the moments of the component couples.*

Let (P,p), (Q,q), (R,r) etc., be the given couples in the same plane, whose respective moments are $P.p$, $Q.q$, $R.r$...... algebraically. Now the couple $(Q.q)$ can be replaced by the couple (Q',p), where $Q'p=Qq$, *i.e.*, by the couple $\left(\frac{Qq}{p}, p\right)$.

In the same way (R, r) can be replaced by $\left(\frac{Rr}{p}, p\right)$.

Similarly for other couples; so that all the couples have the same arm p. Thus all the given couples are equivalent to a single couple whose arm is p, and whose component forces are each equal to $\left(P + \frac{Qq}{p} + \frac{Rr}{p} + \cdots\right)$

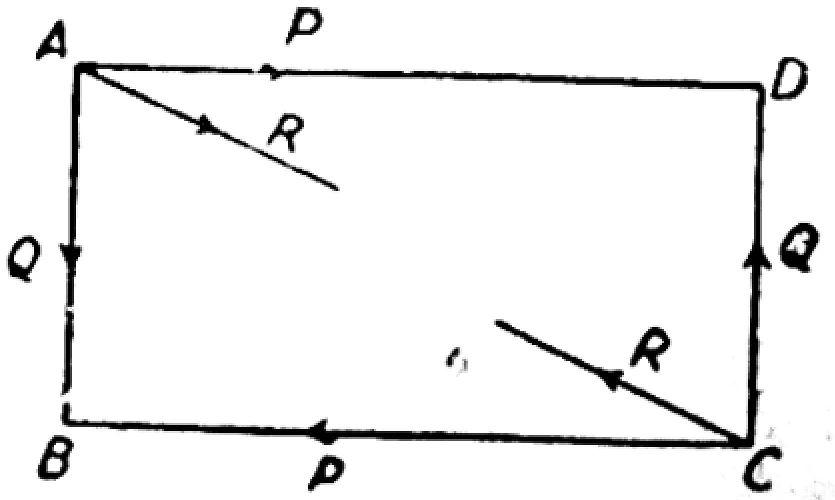
Hence the moment of the resultant couple is

$$Pp + Qq + Rr + \ldots\ldots$$

i.e., the algebraic sum of the moments of the given couples.

Cor. Since a couple is equivalent to another couple of equal moment in a parallel plane, *all the couples which lie in parallel planes can be compounded into a single couple, exactly in the same way as co-planar couples.*

61. Example. *ABCD is a rectangle such that $AB=CD=a$ and $BC=DA=b$. Forces P act along AD and CB, and forces Q act along AB and CD. Prove that the perpendicular distance between the resultant of the forces P, Q at A and the resultant of the forces P, Q at C is* $\frac{Pa-Qb}{\sqrt{P^2+Q^2}}$. (*P. U. 1929*).



The resultant **R at A**

$$=\sqrt{P^2+Q^2}$$

= the resultant R at C.

If d be the perpendicular distance between the two resultants, the moment of the couple (R, d), formed by them must be equal to the sum of the moments of the couple (P, a) and (Q, b).

Hence $Rd = Pa - Qb$.

$$\therefore d = \frac{Pa-Qb}{R} = \frac{Pa-Qb}{\sqrt{P^2+Q^2}}.$$

Exercises VIII

1. ABCD is a square of side 2 ft. Along the sides BA and BC act forces of 24 lbs. and 4 lbs., and along the diagonals AC and BD act forces of $10\sqrt{2}$ and $14\sqrt{2}$ lbs., respectively. Show that the resultant is a couple of moment -20 ft. pounds, the square being drawn positively.

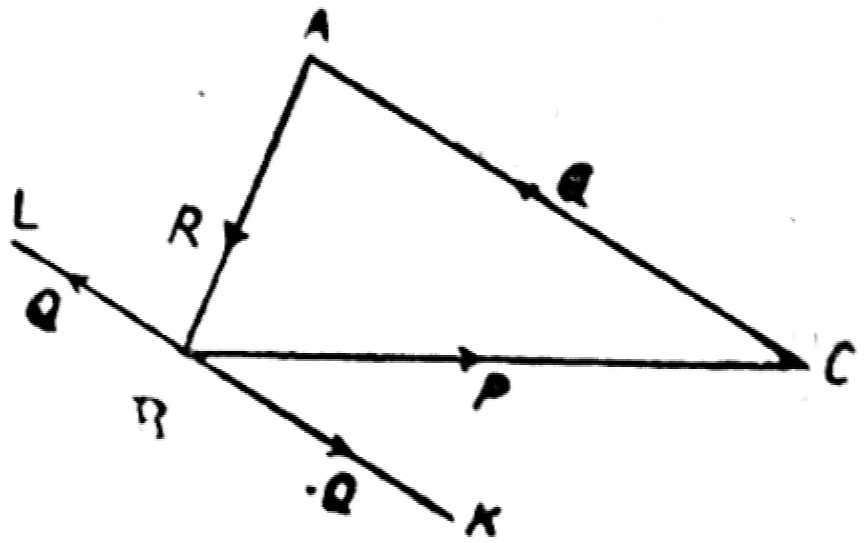
2. ABCD is a rectangle such that AB = 4 ft. and BC = 2 ft. Along AC, BD, DC, AD act forces of $15\sqrt{5}$, $20\sqrt{5}$, P and Q lbs., respectively. Determine P and Q so that the system may reduce to a couple, and find its moment.

3. In a regular hexagon ABCDEF, forces of 1, 2, 3, 1, 2, 3 lbs. act along the sides AB, BC, CD, etc., taken in order, and forces of 4 lbs. each, act along CA and FD. Find the forces which should act along EC and BF, in order to keep the system to equilibrium.

62. *If three forces act along the sides of a triangle, taken in order and be represented by these sides, they are equivalent to a couple, whose moment is represented by twice the area of the triangle.*

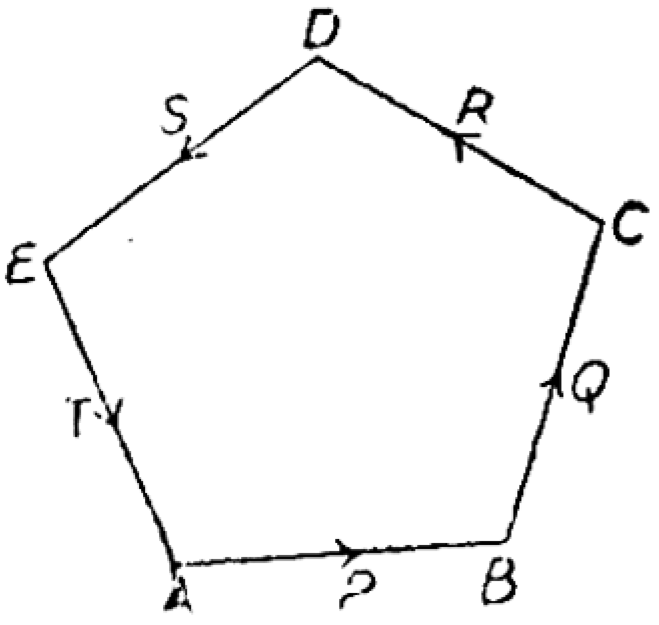
Let the three forces P, Q, R act along the sides of the triangle ABC and be represented by BC, CA, and AB, respectively.

Introduce at B two opposite forces, each equal to Q parallel to AC along BK and BL.



Now the forces P, R and Q along BL, act at the point B and are represented by the sides of the triangle ABC, taken in order. Hence by "the Triangle of Forces," they are in equilibrium. Now we are left with the forces Q, −Q only, which form a couple. The moment of this couple is equal to the algebraic sum of the moments of the forces of the couple about B. Now the moment of −Q is zero and the moment of Q is measured by $AC \times p$ (where p is the perpendicular from B on AC), *i.e.*, by twice the area of the triangle ABC. Hence the moment of the couple, to which the given forces are equivalent, is represented by twice the area of the triangle. This theorem is capable of extension to a polygon as is shown below.

63. *If a system of forces acting in one plane on a rigid body is represented in magnitude, direction and line of action by the sides of a polygon taken in order, the forces are equivalent to a couple whose moment is represented by twice the area of the polygon.*



Let the sides AB, BC, CD, DE, EA, represent forces P, Q, R, S, T in all respects. At A introduce two equal and opposite forces each equal to Q (represented by BC); the effect of the force Q will be unaltered by this process.

Hence Q along BC≡Q at A in the direction BC.

+ a couple whose moment is $2\triangle ABC$. Similarly, R along CD≡R at A in the direction CD+ a couple whose moment is $2\triangle ACD$.

and so on,

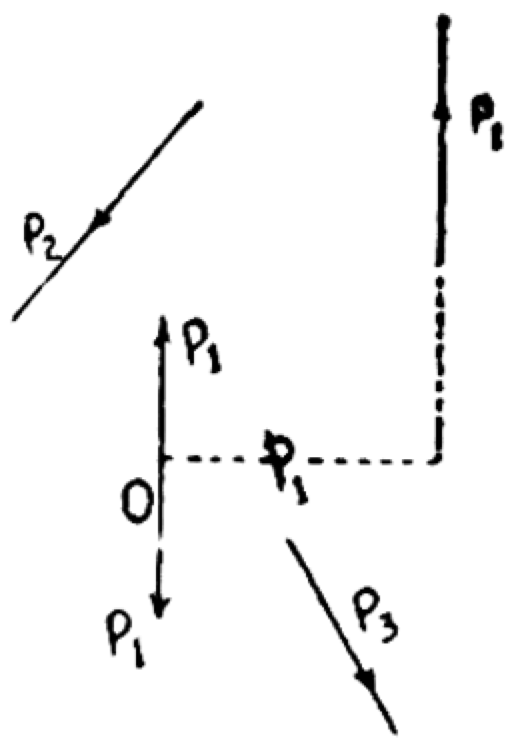
∴ The given forces≡P.Q, R, S, T acting A in the direction indicated by arrowheads *plus* a number of couples whose moments are $2\triangle ABC$, $2\triangle ACD$, 2ADE etc.

=zero (*by poygon of forces*) *plus* a couple whose moment is is equal to algebraic sum of $2\triangle ABC$ $2\triangle ACD$ $2\triangle ADE$ etc.

=A couple of moment equal to twice the area of the polygon.

64. *Any number of co-planar forces acting on a rigid body is equivalent to a force acting at an arbitrary point together with a couple.*

Let P_1, P_2 P_3......be any number of forces and O any arbitrarily selected point in the plane.



By introducing two opposite forces at O, each equal and parallel to P_1, the force P_1 may be replaced by a parallel force P_1 at O, having the same sense, and a couple, whose arm p_1 is perpendicular from O on the given force P_1. (see Art. 57.) Proceeding in this way with the other given forces, we obtain forces P_1, P_2, P_3,...all acting at O, and an equal number of couples.

Since all the forces at O can be compounded into a single force and all the couples into a single couple, *the system of given forces reduces to a single force at O and a single couple, whose moment is the algebraic sum of the moments of the forces about O.*

It ıs impostant to consider the three special cases that arise according as R is or is not zero :—

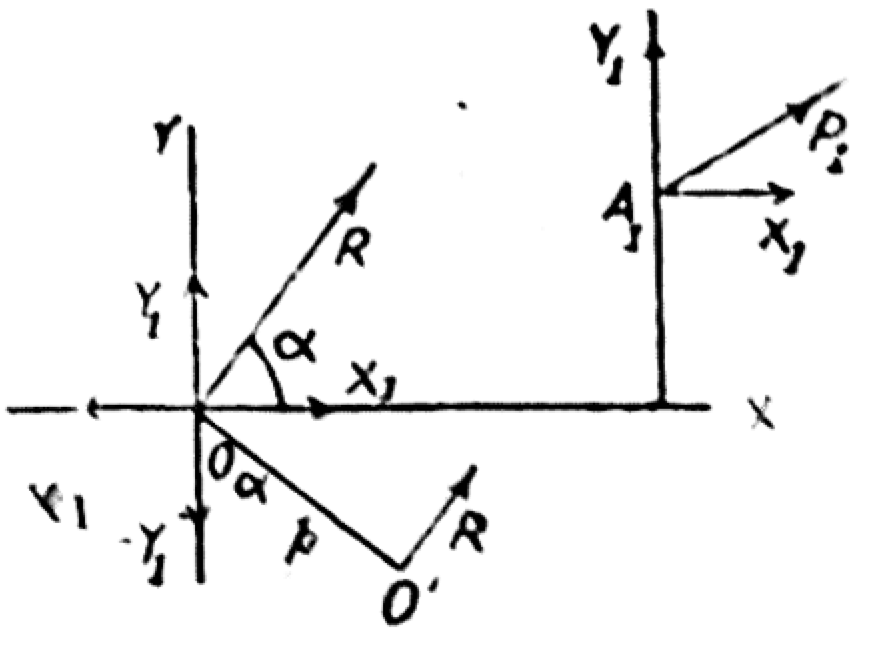
(*i*) If $R=0$, the system of forces reduces to a single couple.

(*ii*) If $R \neq 0$ but $G=0$, the system reduces to the single force R.

(*iii*) If neither R nor G vanishes, the two together will be equivalent to a single force, whose magnitude is R, but which has a different line of action, parallel to its original direction.

Hence we deduce that **any system of forces** (not in equilibrium) **acting in one plane upon a rigid body reduces either to a single force or to a single couple ?**

***The analytical expressions** for the single force and the single couple may be obtained as follows :—



Choosing the arbitrary point O as the origin and any perpendicular lines through it as the axes, let the co-ordinates of A_1, the point of application of the force P_1, be (x_1, y_1). Let X_1, Y_1 be the resolved parts of P_1 parallel to the axes. Now introducing at O along OX two forces X_1 and $-X_1$, and along OY, forces Y_1

and $-Y_1$, we replace P_1 by forces X_1, Y_1 at O, and a couple of moment $x_1Y_1-y_1X_1$.

Proceeding in this way with the other forces $P_2, P_3 \dots$ acting at (x_2, y_2), $(x_3, y_3)\dots$ we obtain the component forces ΣX_1 and ΣY_1 acting along the axes and the couple G of moment $\Sigma(x_1Y_1-y_1X_1)$.

The forces ΣX_1 and ΣY_1 may be compounded into a single force R. We thus have the single force,

$$R=\sqrt{(\Sigma X_1)^2+(\Sigma Y_1)^2}, \quad \dots (1)$$

and the single couple $G=\Sigma(x_1Y_1-y_1X_1)$. $\quad \dots (2)$

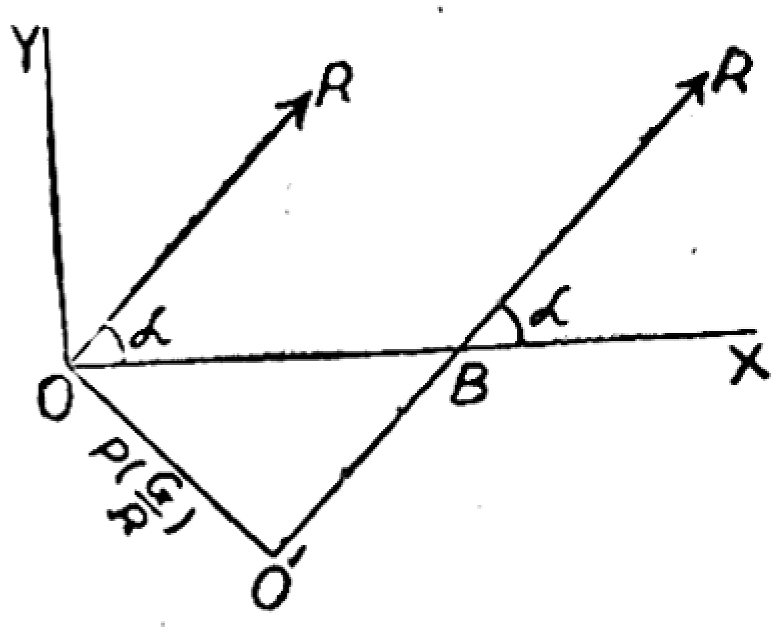
If ΣX_1 and ΣY_1 do not separately vanish, R cannot vanish. In that case the force R and the couple G may be replaced by **a single resultant R**, parallel to R at O. Now we proceed to obtain the *equation of this single resultant R*.

If $OO'(=p)$ be the perpendicular from O on this resultant,

$$\text{then } p=\frac{G}{R} \qquad \text{[See Case I Art. 57].}$$

Now if R makes an angle α with OX, $\Sigma X_1=R\cos\alpha$

$\Sigma Y_1=R\sin\alpha$, so that $\tan\alpha=\dfrac{\Sigma Y_1}{\Sigma X_1}$.



Let the resultant meet the axis of x at B. Then O′B is the line of action of the resultant whose equation is required.

Now, $OB\sin\alpha=OO'$

$\therefore OB=OO'\operatorname{cosec}\alpha=p\operatorname{cosec}\alpha=\dfrac{G}{R}\operatorname{cosec}\alpha=\dfrac{G}{R}\times\dfrac{R}{\Sigma Y_1}=\dfrac{G}{\Sigma Y_1}$.

The co-ordinates of B are, therefore

$$\left(\frac{G}{\Sigma Y_1}, 0\right)$$

Hence the equation of the line of action of the resultant (*i.e.* of O′B) which passes through $\left(\dfrac{G}{\Sigma Y_1}, 0\right)$ and whose slope is $\tan\alpha$ is $y-0=\tan\alpha\left(x-\dfrac{G}{\Sigma Y_1}\right)$

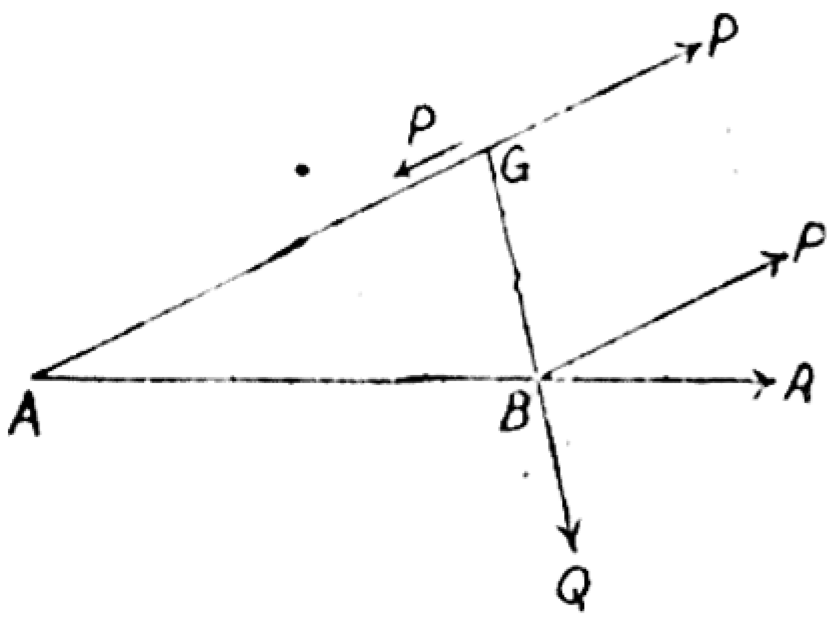
$$= \frac{\Sigma Y_1}{\Sigma X_1}\left(x - \frac{G}{\Sigma Y_1}\right)$$

i.e. $y\Sigma X_1 - x\Sigma Y_1 + G = 0$,

or $x\Sigma Y_1 - y\,\Sigma X_1 - G = 0$,

or $x\Sigma Y_1 - y\Sigma X_1 - \Sigma(x_1 Y_1 - y_1 X_1) = 0$.

65. Ex. 1. Show that the force R represented completely by the side AB of a triangle ABC, can be replaced by forces acting along and represented by AC and CB, respectively together with a couple of moment equal to twice the area of the triangle.

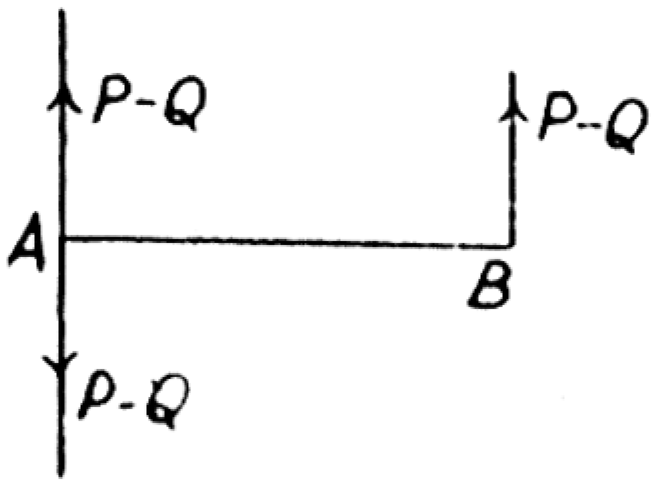


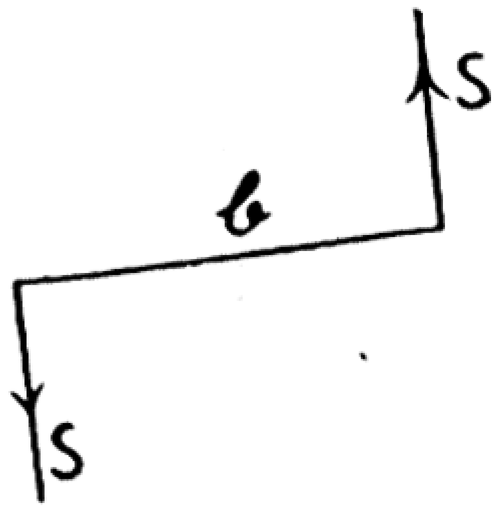
By the Triangle of Forces, the force R is equivalent to forces P and Q, acting at B, but represented in magnitude and direction by AC and CB, respectively.

Now introduce at C two equal and opposite forces P. The force P at B and the force P at C, along CA, give rise to a couple of moment equal to twice the area of the triangle. The other forces left are the force P represented by AC and the force Q represented by CB.

Note.—When we say in vector-notation, $AB = \overline{AC} + BC$, these *lines of action* of the component forces are *not* AC and CB. If AC and CB are to represent P and Q *completely*, *a couple* in addition *is necessary* so that the system may be equivalent to the force R.

Ex. 2. *Two unlike parallel forces P, Q act on a rigid body. If two equal and opposite forces S in any two parallel lines at a distante b apart are combined with them, show that the resultant is displaced by a distance bS/(P−Q).* **(P. U. 1927).**





Let P, Q have the resultant $P-Q$ at A. Draw AB perpendicular to $P-Q$ such that $AB=\dfrac{bS}{P-Q}$. The couple (S, b) can now be replaced by the couple ($P-Q$, AB). The resultant $P-Q$ combined with the couple ($P-Q$, AB) gives the resultant $P-Q$ **at B**. The resultant $P-Q$ is thus displaced by the distance AB, *i.e.* $\dfrac{bS}{P-Q}$.

Exercises VIII A

1. Prove that if three co-planar forces, no two of which are parallel, and which do not meet in a point, are equivalent to a couple, they can be represented in magnitude and direction by the sides of a triangle drawn parallel to their lines of action.

2. X, Y, Z are points on the sides BC, CA, AB of a triangle ABC, such that $BX : XC = CY : YA = AZ : ZB = p : q$. Prove that the forces completely represented by AX, BY, CZ reduce to a couple whose moment is $\dfrac{2\triangle\,(p-q)}{p+q}$, where $\triangle$ is the area of the triangle ABC. (*P. U. 1943*).

3. Points P, Q, R are taken on the sides BC, CA, AB of a triangle dividing each in the same ratio. $1+\lambda.\ 1-\lambda$, round the triangle. Prove that the forces acting along and represented by AP, BQ, CR are equivalent to a couple. Find the moment of the couple. (*P. U. 1926*)

4. Six co-planar forces act on a body along the sides AB, BC, CD, DE, EF, FA of a regular hexagon ABCDEF, in which AC is 1 ft. long. Their magnitudes are 10, 20, 30, 40, P and Q lbs. respectively. Find P and Q so that the system reduces to a pure couple; and show that the moment of the couple is 75 footpounds. (*P. U. 1930*)

Hint. Take two perpendicular lines through the centre as axes, say, FC as x-axis. etc. Sum of the resolved parts along each of the axes must vanish.

5. The moments of a system of co-planar forces acting on a rigid body (not in equilibrium) about three collinear points A, B, C in the plane are G_1, G_2, G_3. Prove that

$G_1.BC+G_2.CA+G_3.AB=0.$ (*P. U. 1939*)

Hint. If the force at A is F along with the couple G_1, $G_2=G_1+F.\ AB \sin \alpha$ and $G_3=G_1+F.AC \sin \alpha$, where α is the angle F makes with AB.

6. OABC is a tetrahedron. Show that the couples represented by the areas OBC, OCA, OAB can have a couple represented by the area ABC as resultant. (*P. U. 1938*)

7. Forces P, 2P respectively act along the sides AB, BC of a rectangle ABCD and a couple of moment P (AD+2AB) acts in the plane of the rectangle in the sense ADCB. If the resultant of the system meets AB, AD in H, K respectively find AH, AK. (*P. U. 1944*)

Hint. The resultant of P and 2P is $\sqrt{5}$ P, acting at B at an angle θ ($=\tan^{-1}2$) with AB.

Taking moments of the forces about H, 2P (HA+AB)−P (AD+2AB)=0.

$\therefore HA=\dfrac{AD}{2}$, and $AK=HA.\ \tan\theta=AD$, *i.e.* K coincides with D.

8. Forces act along the sides AB, BC, CD, DA of a plane quadrilateral, taken in order, and their magnitudes are p, q, r, s times the lengths of the sides in which they act. Prove that they are equivalent to a couple if $(p-q)\ OB=(r-s)\ OD$, and $(q-r)\ OC=(s-p)\ OA$, where O is the intersection of AC and BD. (*D. U. 1943.*)

Hint. As in Ex. 1. Art. 65 the force-p. AB can be replaced by forces p. AO and p. OB acting along AO and OB respectively, together with a couple of moment P times twice $\triangle$ ABO. Proceeding in this way, a couple of moment 2 $(p.\triangle ABO+q.\triangle BCO+r\triangle CDO+s.\ \triangle DAO)$ along with forces in the diagonals AC and BD are obtained. These latter forces balance under the given conditions.

CHAPTER VI

EQUILIBRIUM OF FORCES ON A RIGID BODY

66. *If three co-planar forces acting on a rigid body keep it in equilibrium, they must be concurrent unless they are all parallel.*

Let the three forces be P, Q, R. If they are not all parallel, let any two of them, say, P and Q meet in O. Hence the resultant of P and Q passes through O. Now this resultant and R must be equal and opposite, since they keep the body in equilibrium. This is possible only when R also passes through O. Hence the three forces are concurrent.

Thus the *equilibrium of three forces acting on a rigid body is reduced to the equilibrium of three forces acting at a point.* Hence the conditions of equilibrium may be found by (1) Lami's Theorem, or (2) by the Converse of the Triangle of Forces, or (3) by resolving the forces along two lines and equating the sum of the resolved parts along each of the lines separately to zero (Art. 32).

67. The equilibrium of three forces acting at a point has already been considered in some detail. The problem of three forces acting on a rigid body will not, therefore, offer any difficulty to the student. If two forces with their directions are given and a third force, at a given point, is required which will keep the body in equilibrium, it is necessary only to join the intersection of the two given forces to the point of application of the third and the direction of the third force is found. The magnitude will be easily found by any one of the methods set forth above in Art. 66. If, on the other hand, the position of equilibrium of the body is required, this can be obtained from the consideration that the three forces on the body must meet in a point.

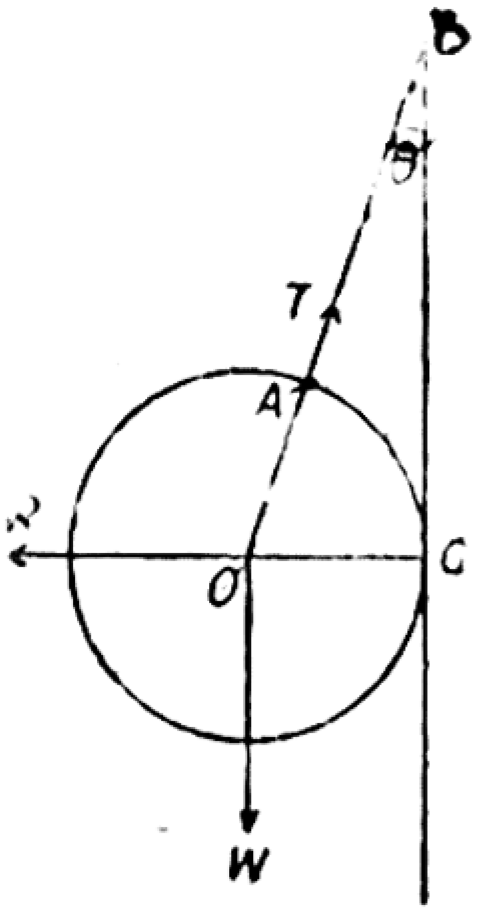
Now it remains only to emphasize that the forces on a body must be taken properly, according to the form in which they may occur. If the force of Tension is exerted by a string, it is obvious that it can only be a *pull*, as distinguished from a *push*. The force of Attraction between two bodies acts along the line of their centres. The Reaction of one smooth body on another must be along the normal to the common surface of contact, there being no tangential action between such bodies. Thus when a smooth sphere or a rod presses against a smooth vertical wall, the reaction must be horizontal; when a rod rests inside a smooth sphere, the reactions being along the normals, intersect at the centre of the sphere. But when the force of reaction acts at a hinge or a joint, it may have any direction, according to the circumstances of the case under consideration.

We proceed to illustrate these points by means of a few examples.

68. Ex. 1. *A sphere hangs by a string, one end of which is fastened to a point on the surface of the sphere and the other to a point in a smooth vertical wall. Find the position of equilibrium of the sphere and the tension of the string.*

Let a be the radius of the sphere and l the length of the string, having its one end at A and the other at B in the wall. Let the sphere rest against the wall at C.

If O is the centre of the sphere, the reaction R of the wall at C must be along the normal CO which is horizontal. Now since the weight and the reaction meet at O, the third force, tension, must also pass through O. Hence OA and AB are in the same straight line. Let the angle OBC be θ.



Then $\sin\theta = \dfrac{a}{a+l}$, ... (1),

which gives the position of equilibrium of the sphere.

By the Converse of the Triangle of Forces, R, T and W are represented by the sides CO, OB and BC of the triangle OBC.

Hence $$\frac{T}{W} = \frac{OB}{BC} = \sec\theta,$$

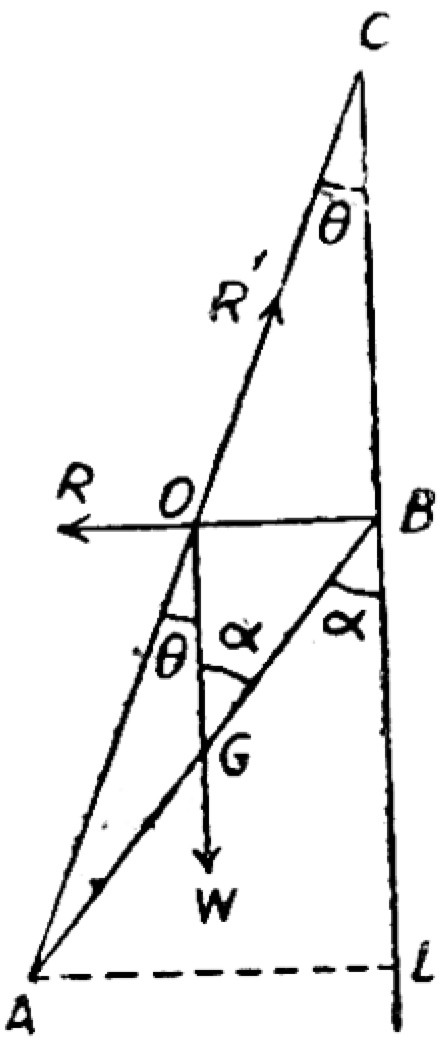
or $$T = W\sec\theta. \qquad \dots (2)$$

Now θ being known from (1), T is found.

Similarly R can be found.

Ex. 2. *A uniform rod AB is movable about a hinge at A and rests with the other end against a smooth vertical wall. If α be the angle which the rod makes with the wall, determine the reactions at the hinge and at B.*

Let the horizontal reaction R at B and the weight W at G, the middle point of AB, meet at O. Then the direction of the reaction at A must be along AO, meeting the wall in C.



Let the angle ACB be θ. We have first to find the relation between α and θ. This may be found by considering that $BL=\frac{1}{2}CL$ (where $AL \perp CL$).

Hence $BL/AL=\frac{1}{2}CL/AL$,

or $\cot\alpha=\frac{1}{2}\cot\theta$. ... (1)

Resolving the forces, $R'\cos\theta=W$,

$\therefore\ R'=W\sec\theta=W\sqrt{1+\frac{1}{4}\tan^2\alpha}$.

Also $R=R'\sin\theta$, etc.

Two useful Trigonometrical formulae.—The relation (1) could be found at once from $\triangle AOB$ by the application of the trigonometrical formula,

$$(m+n)\cot\alpha=m\cot AOG-n\cot BOG. \qquad \text{... (A)}$$

where G divides AB such that $AG:GB::m:n$.

Here this ratio is 1, and the formula gives $2\cot\alpha=\cot\theta$.

The formula (A) can be established as below:—

Let $\angle BOG=\phi$, then $\dfrac{m}{n}=\dfrac{AG}{GB}=\dfrac{AG/GO}{GB/GO}=\dfrac{\sin\theta/\sin(\alpha-\theta)}{\sin\phi/\sin(\alpha+\phi)}$

$$=\frac{\sin(\alpha+\phi)/\sin\phi}{\sin(\alpha-\theta)/\sin\theta}=\frac{\sin\alpha\cot\phi+\cos\alpha}{\sin\alpha\cot\theta-\cos\alpha}.$$

whence $(m+n)\cos\alpha=\sin\alpha\,(m\cot\theta-n\cot\phi)$,

or $(m+n)\cot\alpha=m\cot\theta-n\cot\phi$.

There is **another similar formula**, which is equally useful:—

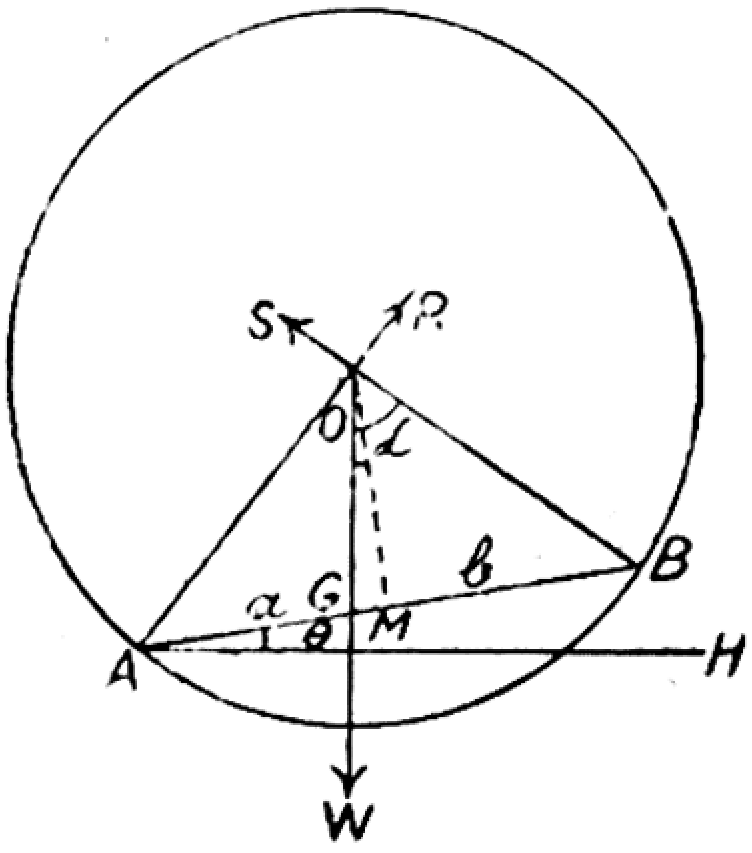
$$(m+n)\cot\alpha=n\cot A-m\cot B. \qquad \text{... (B)}$$

For proving (B), $\dfrac{m}{n}=\dfrac{AG/GO}{GB/GO}=\dfrac{\sin(\alpha-A)/\sin A}{\sin(B+\alpha)/\sin B}$

$$=\frac{\sin\alpha\cot A-\cos\alpha}{\cos\alpha+\sin\alpha\cot B} \text{ etc.}$$

Ex. 3. *A heavy rod whose centre of gravity divides it in the ratio $a:b$, is placed inside a smooth sphere; find the position of equilibrium of the rod, given that 2α is the angle subtended by it at the centre of the sphere.*

Let the rod AB in the position of equilibrium make the angle θ with the horizontal.



The reactions R and S at A and B, respectively, pass through the centre O. The vertical through G, therefore, also passes through O.

Applying the cotangent formula (B), Ex. 2, we get immediately.

$$(a+b)\cot\left(\frac{\pi}{2}-\theta\right)$$

$$=b\cot A-a\cot B$$
$$=(b-a)\cot B$$

$$\therefore \tan\theta=\frac{b-a}{b+a}\cot B$$

$$=\frac{b-a}{b+a}\tan\alpha:$$

The reactions R and S can be found easily by Lami's Theorem.

Note 1. From the above, we can easily deduce that

$$\sin\theta=\frac{b-a}{2\sqrt{r^2-ab}}$$

where r is the radius of the circle. For, we have

$$\tan\theta=\frac{b-a}{(b+a)\cot\alpha}$$

$$\therefore \sin\theta=\frac{b-a}{\sqrt{(b-a)^2+(b+a)^2\cot^2\alpha}}$$

$$=\frac{b-a}{\sqrt{(b-a)^2+(b+a)^2(\operatorname{cosec}^2\alpha-1)}}$$

$$=\frac{b-a}{\sqrt{(b-a)^2-(b+a)^3+(b+a)^2\operatorname{cosec}^2\alpha}}$$

$$=\frac{b-a}{\sqrt{-4ab+(b+a)^3\times\dfrac{4r^3}{(a+b)^2}}}$$

$$=\frac{b-a}{2\sqrt{r^2-ab}}\left[\because \operatorname{cosec}\alpha=\frac{r}{(a+b)/2}\text{ from the figure}\right]$$

Note 2. Applying Lami's Theorem at O,

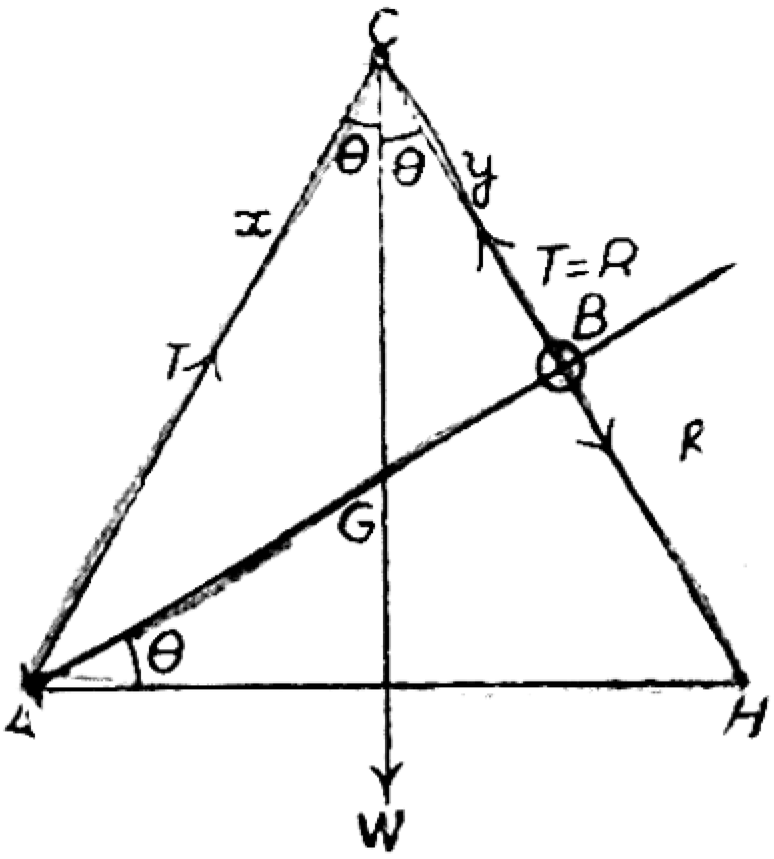
$$\frac{W}{\sin 2\alpha}=\frac{R}{\sin(\alpha+\theta)}=\frac{S}{\sin(\alpha-\theta)}.$$

Since α, θ are both known, the pressures R and S can be calculated in terms of W.

Ex. 4. *One end of a string is fixed to the extremity of a smooth uniform rod, and the other to a ring without weight which slides on the rod, and the string is hung over a smooth rail; prove that if θ be the inclination of the rod to the horizon in the position of equilibrium, 2l the length of the string and 2 a the length of the rod,*

$$l(\tan^3\theta+\tan\theta)=a.$$

Since the string ACB passes over the smooth rail at C, the tensions in the two parts are equal, hence the weight W bisects the angle ACB.



So if AC$=x$, CB$=y$,

$$\frac{x}{y}=\frac{a}{b}, \qquad \dots (1)$$

where GB$=b$ (suppose).

Since the rod is smooth, the reaction R of the rod on the ring is perpendicular to AB; hence the equilibrium of the ring gives T=R, and $\angle$ABC is a right angle.

$\therefore$ $\angle$BCG = $\angle$BAH$=\theta$.

Now applying the cotangent formula,

$$(a+b)\cot\left(\frac{\pi}{2}-\theta\right)=(a-b)\cot\theta.$$

$$\therefore \qquad \tan^2\theta=\frac{a-b}{a+b} \qquad \dots (2)$$

Again, $y=b\cot\theta$ [from rt.-angled $\triangle$ BCG]

$\therefore$ $x=a\cot\theta$. [from (1)]

But $x+y=2l$ $\therefore$ $y=2l-x=2l-a\cot\theta$.

From (2), we have

$$\tan^2\theta=\frac{x-y}{x+y}=\frac{a\cot\theta- \quad l-a\cot\theta)}{2l}$$

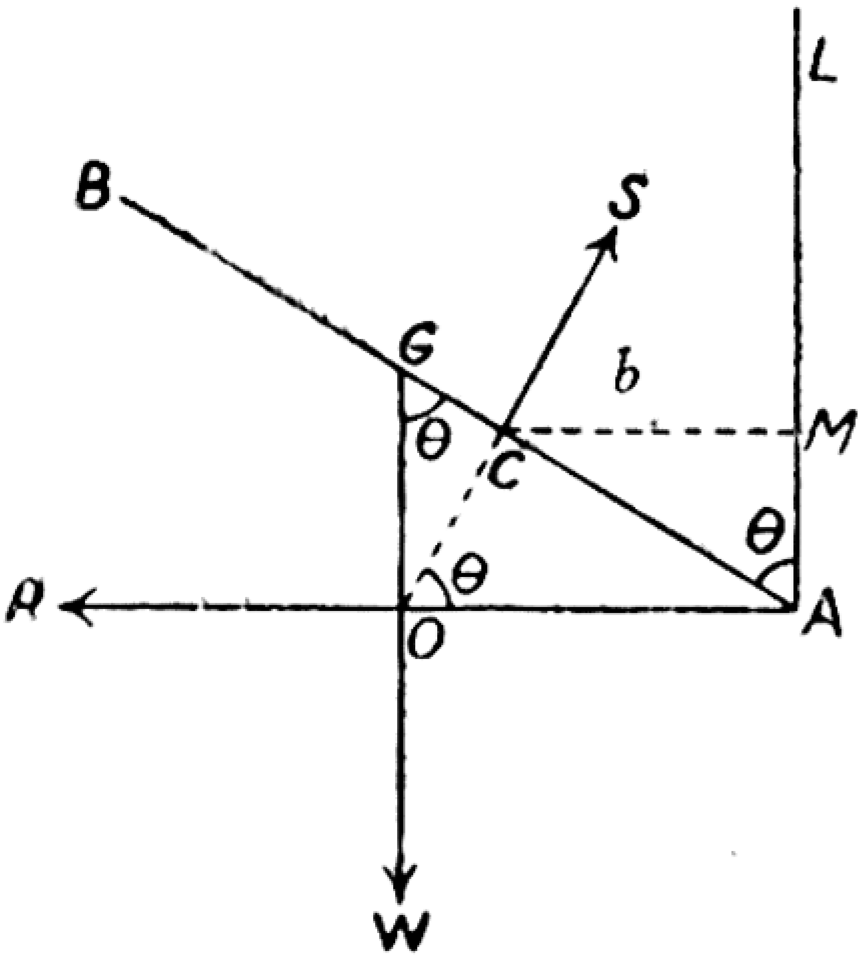
$\therefore$ $2l\tan^2\theta+2l=2a\cot\theta$.

$$\therefore \frac{a}{l} = \frac{\tan^2\theta+1}{\cot\theta} = \tan\theta\,(\tan^2\theta+1).$$

Ex. 5. *A uniform beam of length 2a, rests in equilibrium, with one end resting against a smooth vertical wall and with a point of its length resting upon a smooth horizontal rod which is parallel to the wall, and at a distance b from it, show that the inclination of the beam to the vertical is*

$$\sin^{-1}\left(\frac{b}{a}\right)^{\frac{1}{3}}.$$

Let AB be the rod resting at C on a smooth horizontal rod at a distance b from the vertical wall. Let the inclination of the rod to the vertical wall AL be θ.



The forces acting on the rod are :

(*i*) The reaction S at C, perpendicular to the rod AB and as well as to the smooth horizontal rod.

(*ii*) The reaction at A, perpendicular to the wall.

(*iii*) The weight of the rod acting vertically downwards at G, where G is supposed to be the C. G. of the rod AB.

Since these three forces keep the rod in equilibrium, they must be coplanar and meet in a point.

It is evident from the figure that

$$\angle GAL = \angle COA = \angle AGO = \theta.$$

$$\text{Now } \sin\theta = \frac{CM}{CA}\ (\triangle CAM).$$

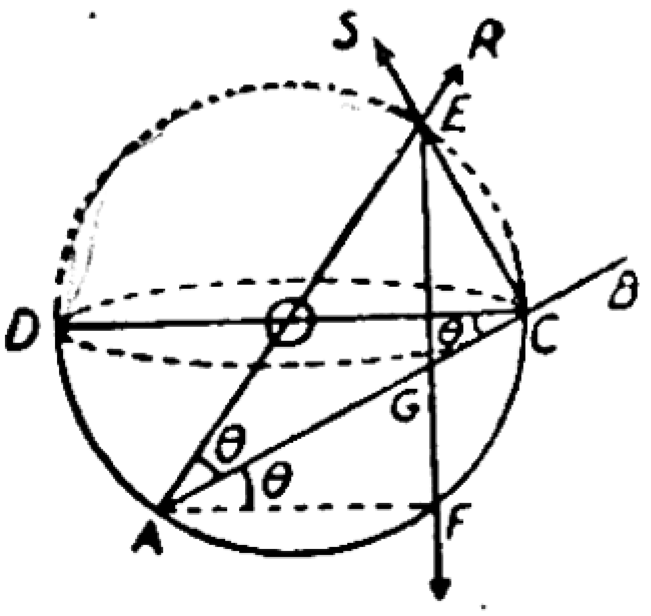
$$\text{Also } \sin\theta = \frac{CA}{OA}\ (\triangle OAC).$$

$$\text{And } \sin\theta = \frac{OA}{AG}\ (\triangle OAG).$$

$$\therefore\ \sin^3\theta = \frac{CM}{CA}\cdot\frac{CA}{OA}\cdot\frac{OA}{AG} = \frac{CM}{AG} = \frac{b}{a}$$

$$\sin\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}},\ \theta = \sin^{-1}\left(\frac{b}{a}\right)^{\frac{1}{3}}.$$

Ex. 6. *A uniform rod rests partly within and partly without a fixed smooth hemispherical bowl, the rim of the bowl being horizontal. Find the position of equilibrium of the rod and the length of the rod when it makes the greatest angle with the horizontal.*



Let CAD be the vertical section of the hemispherical bowl through the central line of the rod; the other half of the section is shown by the dotted line. Let AB be the rod, touching the rim at C.

The reaction R at A passes through the centre O and meets the circular section at E. The reaction S of the rim on the rod is along the perpendicular CE to the rod. (AE is a diameter and ACE is a right angle).

It should be noted that at C the rod is *bodily* being acted upon by the rim, while at A only the end is touching the sphere. Also CE is perpendicular to the tangential line at C to the rim.

Since the third force is the weight of the rod, the vertical through E passes through G, the middle point of AB, meeting the sphere at F. Now AF is obviously the horizontal line through A.

Let $\angle CAF = \theta$, then $\angle OCA = \theta = \angle OAC$,
and $\angle EAF = 2\theta$.

If $AB = 2a$ and radius $= r$,
then $AF = a \cos \theta$, and also $= 2r \cos 2\theta$, from triangle AEF.

Hence
$$a \cos \theta = 2r \cos 2\theta \qquad \ldots(1)$$
gives the position of equilibrium of the rod.

Arranging (1) as a quadratic equation in $\cos \theta$,
$$4r \cos^2\theta - a \cos \theta - 2r = 0,$$
whence
$$\cos\theta = (a \pm \sqrt{a^2 + 32r^2})/8r. \qquad \ldots(2)$$

Since $\cos \theta$ is $+ve.$, the positive sign must be taken before the radical.

Hence as a decreases, $\cos \theta$ decreases and θ increases.

The angle is, therefore, the greatest when the rod has the least length, which is AC. In that case
$$a = r \cos \theta. \qquad \ldots(3)$$

Hence from (2) and (3), $2r^2=3a^2$

i. e., the angle θ is greatest when $\cos\theta=\frac{\sqrt{2}}{\sqrt{3}}$,

$$\left(\text{or } \sin\theta=\frac{1}{\sqrt{3}}\right)$$

and from (3), the length of the rod, in that position, $=\frac{2\sqrt{2}}{\sqrt{3}}r$.

The *maximum length of the rod* is found by putting $\theta=0$ in (1), which gives $AB=4r$, a result which is obvious, otherwise.

Exercises IX

1. A uniform ladder of weight 16 lbs. leans against a perfectly smooth wall. Find the thrusts which it exerts against the wall and the ground when the ladder is 20 ft. long and reaches a height of 16 ft.

2. A uniform rod AB of weight $\frac{1}{4}$ cwt. hinged at A, is supported in a horizontal position by a rope attached to B and making an angle of 45° with the rod. Find the tension in the rope and the force at the hinge.

3. A uniform rod of length 12 ft. and weight 20 lbs. is placed over a smooth peg so that one end rests against a smooth vertical wall. The distance of the peg from the wall is 9 inches. Find the position of equilibrium and the force of pressure on the peg.

4. A uniform rod of length 15 inches and weight 12 lbs. is suspended by two strings, 9 and 12 inches long, respectively, which are attached to the ends of the rod and also to a nail above the rod. Find the position of equilibrium and the tension of each string.

5. A weightless rod ACB can turn freely in the vertical plane about the fixed point C, the distances AC and CB being 6 ft. and 5 ft., respectively. A mass of 30 lbs. is suspended from B, and the rod is kept in the horizontal position by a light string attached to A and inclined at 30° to the horizontal. Find the tension of the string and the force of constraint at C.

6. A uniform rod AB of mass 20 lbs. can turn freely about the extremity A which is fixed and rests in a position inclined at an angle of 60° to the vertical, being supported by a horizontal force applied at the lower end B. Find the magnitude of the applied force and of the reaction at A.

7. If in Q. 6 above, the applied force be inclined at 60° to the horizontal, find the force and the reaction.

8. A sphere of weight 6 lbs. rests on two smooth inclined planes of inclination 30°, 60°, whose common section is a horizontal straight line; find the pressure on each plane.

9. A heavy equilateral triangle, suspended from a smooth peg by a string, the ends of which are attached to two of its angular points, rests with one of its sides vertical; show that the length of the string is double the altitude of the triangle.

10. A triangular lamina ABC is suspended from a point O by light strings fastened to the points A, B and hangs so that BC is vertical. Prove that if α, β be the angles which the strings AO, BO make with the vertical, then $2 \cot \alpha - \cot \beta = 3 \cot B$. [*P.U. 1936*].

11. A circular disc is kept at rest by three forces acting perpendicularly to the circumference at three given points therein. Show that the forces are as the sides of the circumscribing triangle, that pass through those points.

12. A weight is suspended from a light rod which passes over two fixed supports, 6 ft. apart. On moving the weight 6 inches nearer to one support, the pressure on that support is increased by 10 lbs., what is the amount of the weight?

13. A plummet of weight 10 lbs., is immersed in a current of water, the string being held in the hand; and the string ultimately settles in a position in which it makes an angle of 30° with the vertical. Draw a diagram, showing the nature of the forces acting on the plummet, and determine the tension of the string.

14. A rod weighing 20 lbs. is placed inside a smooth sphere and subtends a right angle at the centre. If its centre of gravity divides it in the ratio $\sqrt{3}-1 : \sqrt{3}+1$, show that the rod makes the angle 30° with the horizontal in the position of equilibrium, and that the reactions are $5(\sqrt{6}+\sqrt{2})$ and $5(\sqrt{6}-\sqrt{2})$ lbs., wt. respectively.

·15. Prove that a uniform rod cannot rest entirely within a smooth hemispherical bowl, except in a horizontal position.

16. If a uniform heavy rod be supported by a string fastened at its ends and passing over a smooth peg, prove that it can rest only in a horizontal or a vertical position.

. 17. A uniform heavy rod of length $2a$ is placed across a smooth horizontal rail and rests with one end against a smooth vertical wall, the distance of which from the rail is h. Show that the angle the rod makes with a horizon is $\cos^{-1}\left(\frac{h}{a}\right)^{\frac{1}{3}}$.

18. Show that no uniform rod can rest partly within and partly without a fixed smooth hemispherical bowl at an inclination to the horizon greater than $\sin^{-1}\frac{1}{\sqrt{3}}$. [See worked Ex. 6].

19. A uniform rod is capable of turning freely in a vertical plane round a hinge at one extremity : a cord attached to the other extremity passes over a smooth horizontal rail (perpendicular to the vertical plane through the rod, and in the same horizontal plane with the hinge) and supports a body of given mass hanging freely ; determine the position of equilibrium of the rod, and direction and magnitude of the reaction at the hinge.

20. Show that the inclination of the rod to the vertical in Ex. 6, Art. 68, cannot be less than 45° and that the least length of the string, for which equilibrium is possible, is a.

21. ACB is a uniform rod of weight W ; it is supported (B being uppermost) with the end A against a *smooth* vertical wall AD by means of a string CD, DB being horizontal and CD inclined to the wall at an angle 30°. Find the tension of the string and the reaction of the wall. [*P.U. 1944*].

22. A heavy uniform beam AB can turn freely about a hinge at A and the end B is attached to a string which passes over a smooth pulley at C vertically above A and carries a weight P hanging freely. Prove that in equilibrium

$$BC = \frac{2P}{W} AC.$$

[*P.U. 1943*].

Hint : AO is the direction of the reaction of the hinge at A, where O is the midpoint of BC. Forces, are parallel and proportional to the sides of the $\triangle$ AOC.

23. A string 9 ft. long has one end attached to the extremity of a smooth uniform heavy rod 2 ft. long, and at the other end carries a weightless ring, which slides on the rod. The rod is suspended by means of the string from a smooth peg ; show that the angle θ which the rod makes with the horizon is given by $\tan\theta = 3^{-\frac{1}{3}} - 3^{-\frac{2}{3}}$.

Conditions of Equilibrium of any number of Co-planar Forces.

69. Equilibrium of Co-planar Forces.—If any number of co-planhr forces act on a rigid body, the system is equivalent to a single force R at an arbitrarily chosen point O, and a couple G, whose moment is equal to the algebraic sum of the moments of the forces about O (Art. 64).

Since a force and a couple cannot balance each other, R and G must vanish separately, in order that the forces may be in equilibrium. Thus the **conditions of equilibrium** are that

$$R = 0 \text{ and } G = 0.$$

Now R=0, *if the sum of the resolved parts of the forces along any two directions, separately, is zero.* (Art. 32). ...(1)

Again, G=0, *if the algebraic sum of the moments of the forces about any point in the plane vanishes.* ..(2)

These two conditions may *analytically* be expressed by *three conditions* as follows, (from Art. 64) :—

$$\Sigma X_1 = 0,\ \Sigma Y_1 = 0, \qquad \text{...(3)}$$

$$\text{and } \Sigma(x_1 Y_1 - y_1 X_1) = 0. \qquad \text{...(4)}$$

Note 1. It should be clearly understood that R is the same in magnitude for any point of reference O, while G would have different values for different positions of O. But when R=0, the couple G has a constant value for any point O. Hence in such a case when G vanishes for one position of O, it vanishes for every other position also in the plane.

Note 2. The above three conditions, namely, $\Sigma X_1 = 0$, $\Sigma Y_1 = 0$ and $\Sigma(x_1 Y_1 - y_1 X_1) = 0$ (*i. e. the algebraic sum of the moments of the forces about any point in their plane should also vanish*) are **both necessary and sufficient.**

(*i*) That they are *necessary conditions* of equilibrium is evident from the fact that if any one of the conditions were suppressed, the equilibrium could not be guaranteed.

(*a*) Suppose only two of the conditions, namely $\Sigma X_1=0$ and $\Sigma Y_1=0$ are satisfied but the third condition is not satisfied, the system of forces would then reduce to a couple.

(*b*) Again suppose that $\Sigma X_1=0$ and $\Sigma(x_1 Y_1-y_1 X_1)=0$ but ΣY_1 is not zero, then the system of forces would reduce to a single force perpendicular to the direction of ΣX_1.

Similar remarks apply when $\Sigma Y_1=0$, $\Sigma(x_1 Y_1-y_1 X_1)=0$, but ΣX_1 is not zero.

(*ii*) These three conditions when simultaneously satisfied are evidently *sufficient* for equilibrium. For, if the system of forces be not in equilibrium, the forces must reduce ultimately either to a single force, or to a single couple (Art. 64).

But the forces cannot reduce to a single force; since the algebraic sum of the resolved parts of the forces along two lines in the plane vanishes (*i. e.* since $\Sigma X_1=0$ as well as $\Sigma Y_1=0$)

Again, the system cannot reduce to a single couple, since the algebraic sum of the moments of the forces about a point in the plane is zero [*i e*, since $\Sigma(y_1 X_1-x_1 Y_1)=0$].

70. Other forms of the conditions of equilibrium.

(A) *Any system of co-planar forces is in equilibrium if the algebraic sum of the moments of the forces about each of any three non-collinear points is zero.*

Let the points be O_1 O_2, O_3. Choosing O_1 as the point of reference to which all the forces of the system are transferred, we get a force R only; for, the couple G has its moment zero. And since the sum of the moments about O_2 vanishes, either $R=0$, or R acts along O_1O_2. Again, since the sum of the moments of the forces about O_3 also vanishes, the moment of R about O_3 vanishes. This is not possible unless $R=0$, because O_1, O_2, O_3 being non-collinear, R cannot pass through O_3. The system is, therefore, in equilibrium.

(B) *The forces are in equilibrium if the sum of their moments about each of any two points O_1, O_2 vanishes separately, and the sum of the resolved parts of the forces in any direction except the one perpendicular to $O_1 O_2$ also vanishes.*

Proceeding as before, either $R=0$, or R acts along $O_1\ O_2$. If R acted along $O_1\ O_2$, it would have a resolved part along the given direction, which is contrary to the hypothesis. Hence $R=0$ and the system is in equilibrium.

71. Recapitulation We have seen that every set of the conditions of equilibrium gives rise to three equations among the forces. These three statical relations combined with other geometrical or analytical relations which may exist, will determine the equilibrium of the body.

The three statical relations are generally found convenient when obtained in the following manner :—

1. *Resolve the forces in any suitable direction.*

2. *Resolve the forces in the perpendicular direction.*

Much simplification may be brought about by choosing the two perpendicular directions in such a way that most of the forces act in one or the other direction.

In practical problems it will be found that the convenient directions are generally, horizontal and veŕtical.

3. *Take moments of the forces about a suitable point.*

This point should be chosen in such a way that most of the forces pass through it, so that the fewest forces appear in the relation.

72. Constrained Bodies. Case (i).—When a body is fixed at one point, as a rod or a lamina by means of a hinge, the body is said *to be constrained.* The **rod whose one point is fixed,** can turn in any plane about that point, but the lamina can turn about the fixed point in its own plane only.

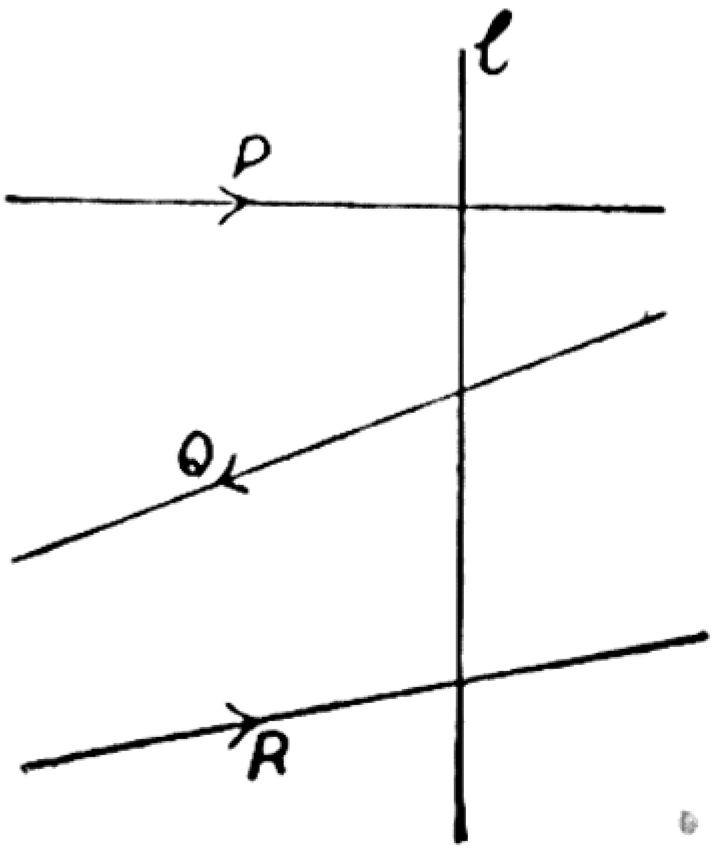
If a number of forces, coplanar with the lamina, act upon it, they can be reduced to a single force R at the hinge and a couple G whose moment is equal to the algebraic sum of the moments of the forces about the hinge. The force R is neutralised by the reaction of the hinge; hence the condition of equilibrium for a constrained lamina is that *the algebraic sum of the moments of the forces about the point of constraint vanishes.*

Case (ii) If a body has *two points fixed*, as a door fixed by two hinges, the body can turn only about *the straight line joining the points as axis*. Now (*a*) if a door be acted upon by a force perpendicular to the plane of the door, its tendency to turn would depend upon the magnitude of the force and the perpendicular distance between the force and the axis; (*b*) but if the force be inclined at any other angle to the door, it can be resolved into two components, one perpendicular to the plane of the door and the other lying in the plane. The component lying in the plane can have obviously no turning effect, and so the turning effect of the force would depend upon its resolved part perpendicular to the plane of the door. For these reasons we have the following definition :—

The moment of a force **about a straight line** *is defined to be the product of the component, perpendicular to the straight line and the shortest distance between the straight line and the line of action of this component.*

Hence for the equilibrium of a body constrained to turn about **an axis** *and acted upon by any forces we have the condition that the algebraic sum of the moments of the forces about the axis vanishes.*

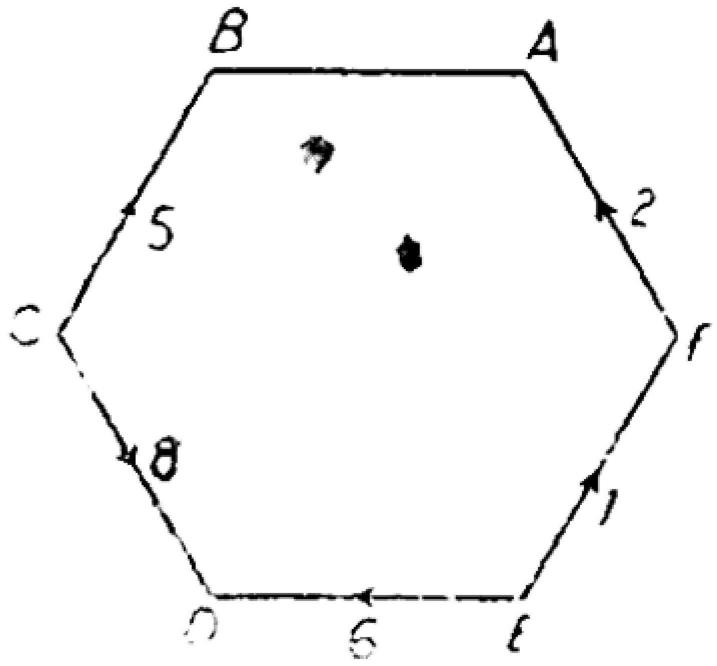
73. **Theorem.—If three forces keep a body in equilibrium, they must lie in the same plane.**



Draw any straight line l meeting the lines of action of the two forces P, Q and not parallel to any third force R, then P and Q (since they meet l) have no moment about l; $\therefore$ R also has no moment about l, otherwise it will have the tendency to turn the body about l. Hence the line of action of R meets l. Since any straight line l which meets the line of action of P, Q and is not parallel to R, meets the line of action of R, the forces P, Q, R, must be co-planar.

Remarks. In solving problems on the equilibrium, of a system of bodies under the action of co-planar forces, we may either treat the system as a whole, ignoring the mutual actions of the parts of which it is constituted, since these actions are equal and opposite; or, we may consider the equilibrium of each part of the system separately, under the action of external forces impressed upon it and of such other parts of the system as act upon it. It may be noted carefully that if a group of bodies is in equilibrium, the external forces acting on the group will form a system of forces in equilibrium. We now give a few illustrative examples.

Ex. 1. *Forces act along the sides of a regular hexagon ABCDEF as follows : along CB, 5 lbs ; CD, 8 lbs.; ED, 6 lbs.; EF, lb. ; FA, 2 lbs. Is the system in equilibrium ?*



Let each side $=a$.

Taking moments about C, D, E, respectively we have

(*i*) (C) $6.a \sin 60^\circ - 1.2\,a \cos 30^\circ - 2.2a \cos 30^\circ$, which vanishes.

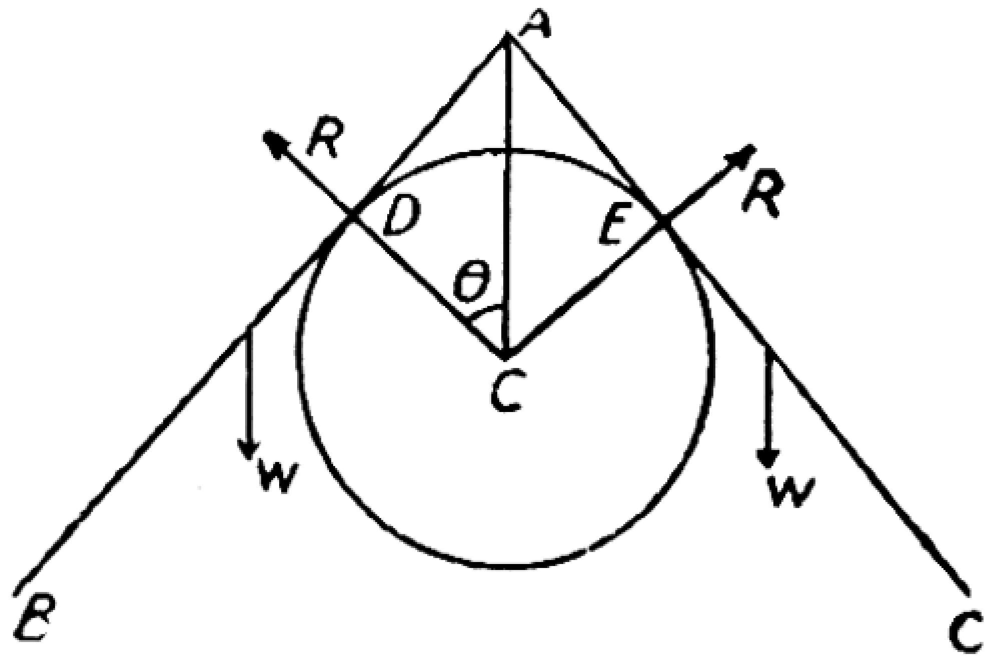
(*ii*) (D) $5.a \sin 60^\circ - 1.a \sin 60^\circ - 2.2a \cos 30^\circ$; this also vanishes.

(*iii*) (E) $5.2a \cos 30^\circ - 8.\,a \sin 60^\circ - 2a \sin 60^\circ$; this also vanishes.

Thus the algebraic sum of the moments of the forces about each of three points is zero. Hence the system is in equilibrium.

The student should obtain the result by the other two methods also.

Ex. 2. *Two equal heavy rods, of weight W and length 2a, are freely hinged together, and placed symmetrically over a smooth fixed sphere of radius r. Show that the inclination of each rod to the horizontal is given by* $r\,(\tan^3\theta + \tan\theta) = a$.



Both the rods are symmetrically situated with respect to the sphere, centre C.

The reactions R at the points of contact D, E of the rods with the sphere are equal and pass through C.

The weight W of each rod acts at its middle point, and is evidently vertical.

Treating the rod as a rigid body, the external forces acting upon it, are the weights W, W, and the reactions R, R. Resolving vertically, we have

$$W = R\cos\theta. \qquad \dots (1)$$

Taking moments about A for the rod AB, we have

$$W.\ a\cos\theta = R.\ AD. \qquad \dots (2)$$

$$\text{Now } AD = r\tan\theta.$$

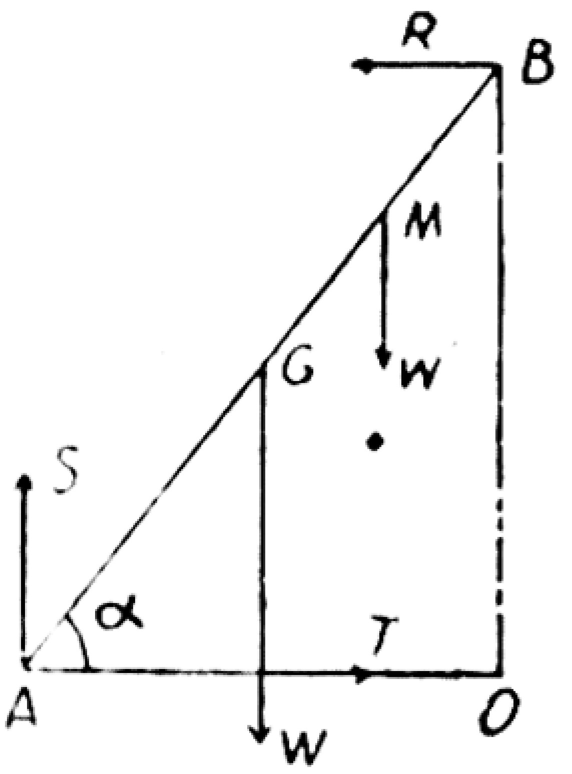
$$\therefore\ a\cos\theta = r\tan\theta\sec\theta$$

i.e., $r\tan\theta\ \sec^2\theta = a$

i.e., $r(\tan^3\theta + \tan\theta) = a.$

Ex. 3. *A ladder rests at an angle* α *to the horizon, with its ends resting on a smooth floor and against a smooth vertical wall the lower end being attached by a string to the junction of the wall and the floor ; find the tension of the string.*

Find also the tension of the string when a man, whose weight is one-half that of the ladder, has ascended the ladder two-thirds of its length.



Let AB be the ladder of length a and G its middle point.

Let T be the tension of the string, R, S. the reactions of the wall and the floor ; W, W′ the weights of the rod and man.

Resolving horizontally and vertically, we have

$$W + W' = S \qquad \dots (1)$$

$$T = R. \qquad \dots (2)$$

Taking moments about A, we have

$$W.\ \frac{a}{2}\cos\alpha + W'.\ \frac{2}{3}a\cos\alpha = R.\ a\sin\alpha,$$

i.e. $3W + 4W' = 6R\tan\alpha. \qquad (3)$

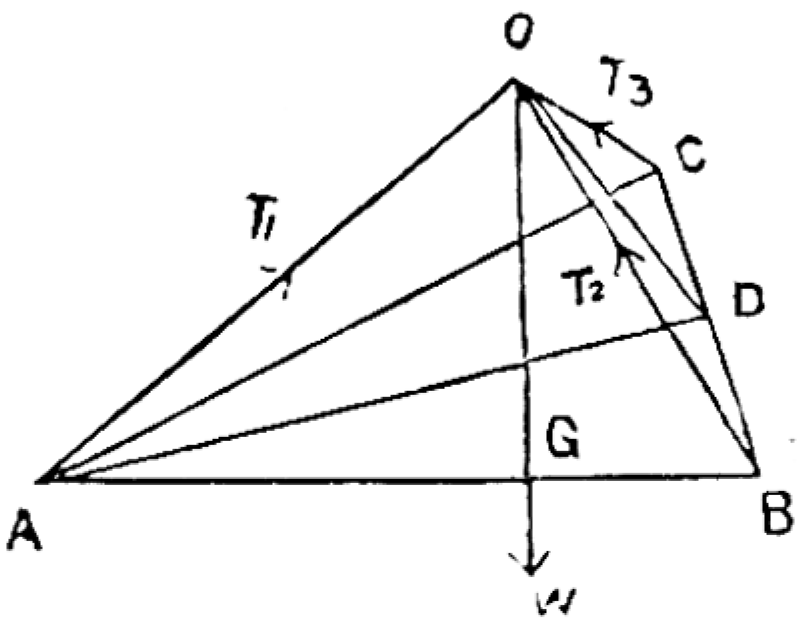
$$\therefore\ R = \frac{1}{6}(3W + 4W')\cot\alpha = T.$$

Since in the 1st case, $W' = 0$, and in the 2nd case, $W' = \frac{1}{3}W$,

$$\therefore\ T = \frac{1}{2}W\cot\alpha \text{ and } T = \frac{5}{6}W\cot\alpha, \text{ respectively.}$$

Ex. 4. *If a triangle is suspended from a fixed point by strings attached to the angles, show that the tension of each string is proportional to its length.*

Assume that the weight W of the triangle ABC acts at its centroid G.



Let T_1, T_2, T_3 be the tensions of the strings OA, OB, OC, O being the point of suspension.

The triangle is under the action of four forces T_1, T_2, T_3, W.

If T be the resultant of T_2, T_3 then the four forces acting on the triangle are reducible to three forces T_1, T, W.

For equilibrium, T_1, T, W must be coplanar. Thus the line of action of T is OD, where D is the midpoint of BC and the line of action of W passes through O.

From Lami's Theorem $\dfrac{T_1}{\sin GOD} = \dfrac{W}{\sin AOD}$.

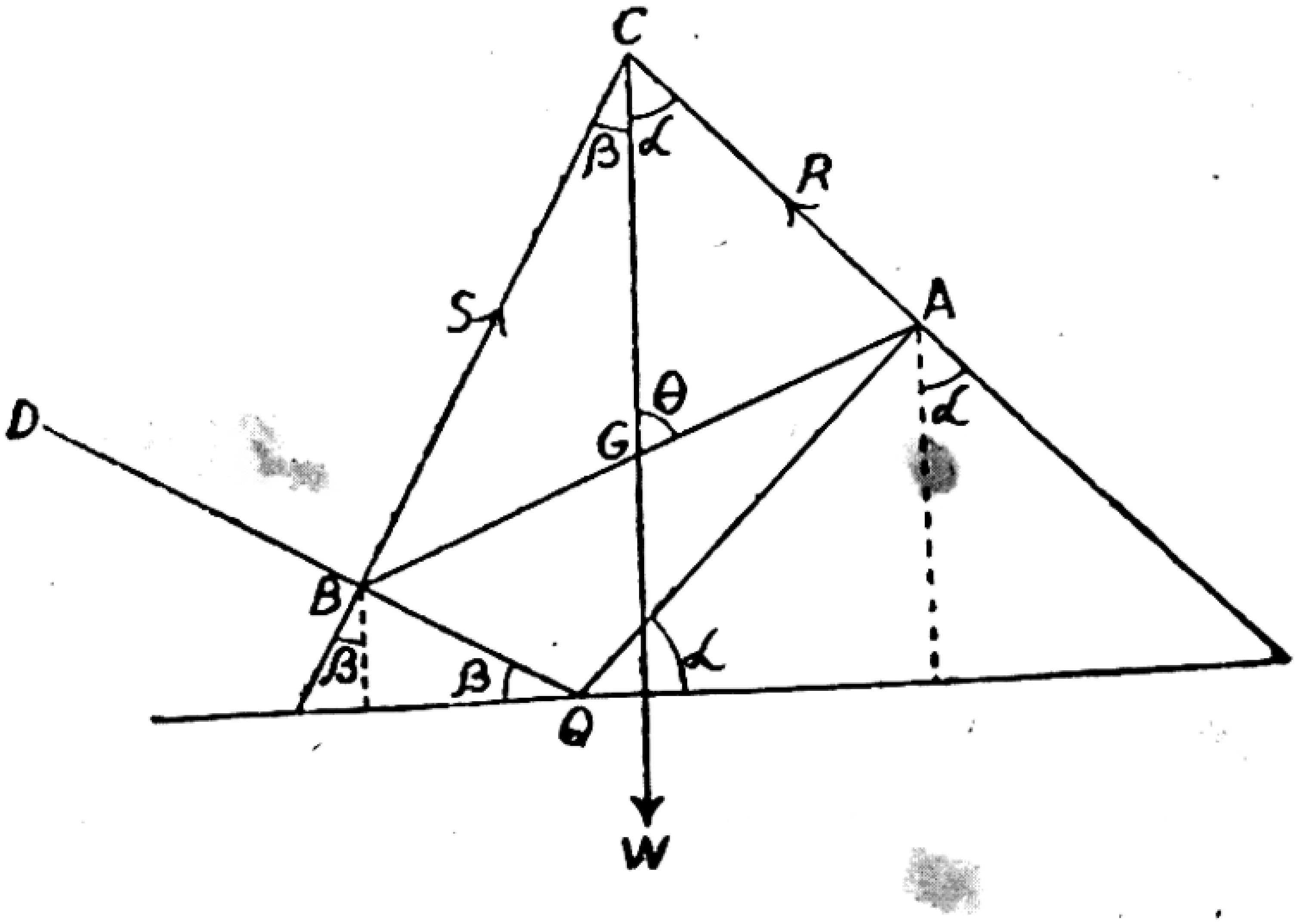
Now from △ GOD, $\dfrac{\sin GOD}{GD} = \dfrac{\sin ODA}{OG}$.

Also $GD = \dfrac{1}{3}$ AD, and $\dfrac{\sin ODA}{\sin AOD} = \dfrac{OA}{AD}$.

$$\therefore \frac{T_1}{OA} = \frac{W}{3\,OG}.$$

Hence $\dfrac{T_1}{OA} = \dfrac{T_2}{OB} = \dfrac{T_3}{OC}$.

Ex. 5. *A uniform rod, of weight W, rests with its ends in contact with two smooth planes, inclined at angles α and β respectively to the horizon, and intersecting in a horizontal line. Find the inclination of the rod to the vertical and also find the reactions at the ends of the rod.*



Let OA and OD be the two planes, and let a rod AB rest on the planes at an inclination θ to the vertical. Since the rod is uniform $\therefore$ the C. G. must lie at its middle point.

The forces acting upon the rod are the reactions, S and R $\perp$ to the planes OB and OA at the points B and A, and the weight W of the rod acting vertically downwards.

These three forces must be co-planar, and should meet in one point C.

In order to find the inclination θ to the vertical we apply the formula,

$$(m+n) \cot \theta = m \cot \alpha - n \cot \beta.$$

In the $\triangle$ ABC, since G is the mid. point of AB we have,

$$(a+a) \cot \theta = a \cot \beta - a \cot \alpha$$

$$2a \cot \theta = a \cot \beta - a \cot \alpha$$

$$\therefore \theta = \cot^{-1}\left(\frac{\cot \beta - \cot \alpha}{2}\right).$$

In order to find the reactions R and S, we apply Lami's Theorem.

Thus $\dfrac{R}{\sin(\pi-\beta)} = \dfrac{S}{\sin(\pi-\alpha)} = \dfrac{W}{\sin(\alpha+\beta)}$ gives the reactions.

Aliter.—Let us apply the three general conditions of equilibrium, *i. e.* $X=0$, $Y=0$, $G=0$.

Resolving horizontally and vertically we have

$$R \sin \alpha = S \sin \beta \quad \text{.....}(i)$$

$$R \cos \alpha + S \cos \beta = W \quad \text{......}(ii)$$

Let us take moments about G.

$$S.\ BG \sin(\theta-\beta) = R.\ AG \sin[\pi-(\theta+\alpha)],$$

or $$S. \sin(\theta-\beta) = R \sin(\alpha+\theta) \quad \text{......}(iii)$$

From (*i*) and (*iii*) we have

$$\frac{R}{S} = \frac{\sin \beta}{\sin \alpha} = \frac{\sin(\theta-\beta)}{\sin(\alpha+\theta)}$$

$$\therefore \quad \sin \alpha \sin(\theta-\beta) = \sin \beta \sin(\alpha+\theta),$$

or $$\sin \alpha (\sin \theta \cos \beta - \cos \theta \sin \beta)$$
$$= \sin \beta (\sin \alpha \cos \theta + \cos \alpha \sin \theta),$$

or $$\sin \alpha \sin \theta \cos \beta - \sin \alpha \cos \theta \sin \beta$$
$$= \sin \beta \sin \alpha \cos \theta + \sin \beta \cos \alpha \sin \theta,$$

$$\text{or } \sin\alpha \sin\theta \cos\beta - \sin\beta \cos\alpha \sin\theta = \sin\alpha \sin\beta \cos\theta.$$

$$\therefore \quad 2\cot\theta = \cot\beta - \cot\alpha,$$

$$\text{or} \qquad \theta = \cot^{-1}\frac{(\cot\beta - \cot\alpha)}{2}.$$

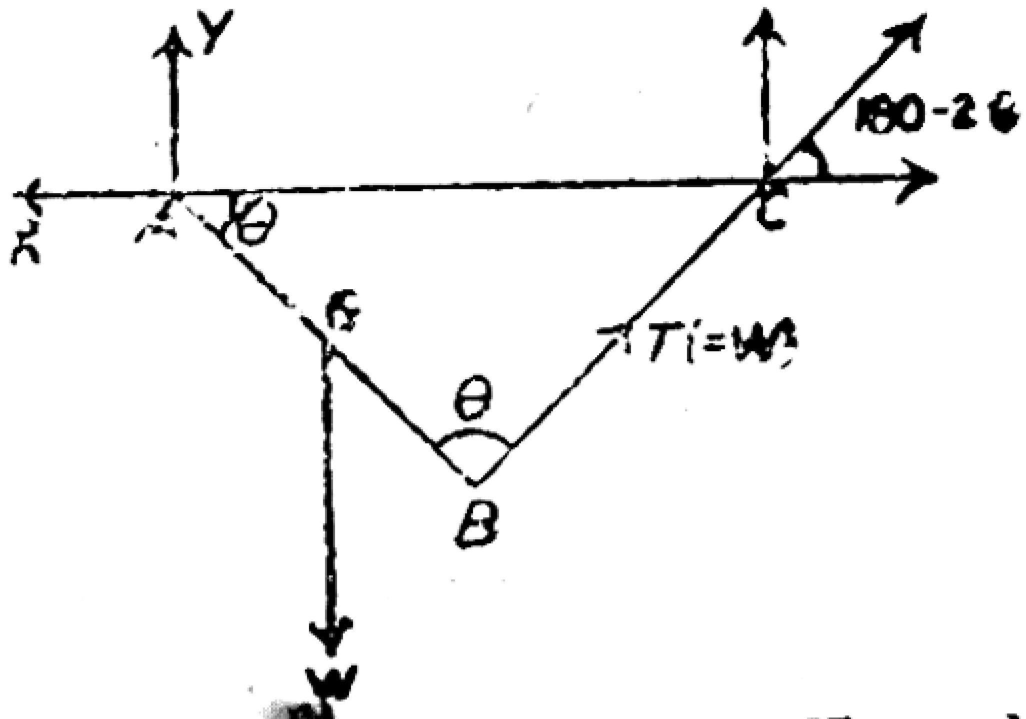
Again from (*i*) and (*ii*)

$$\frac{R}{\sin\beta} = \frac{S}{\sin\alpha} = \frac{R\cos\alpha + S\cos\beta}{(\cos\alpha\sin\beta + \sin\alpha\cos\beta)}.$$

$$= \frac{W}{\sin(\alpha+\beta)}.$$

$$\therefore \quad R = \frac{W\sin\beta}{\sin(\alpha+\beta)}, \qquad S = \frac{W\sin(\alpha+\beta)}{\sin\alpha}.$$

Ex. 6. *A uniform rod has its upper end fixed to a hinge and its other end attached by a string to a fixed point in the same horizontal plane, as the hinge, the length of the string being equal to the distance between the fixed point and the hinge. If the tension of the string be equal to the weight W of the rod, show that the rod is inclined to the horizontal at an angle* $\tan^{-1}\frac{1}{2}$, *and that the action at the hinge is equal to a force* $\frac{W}{5}\sqrt{10}$ *inclined at an angle* $\tan^{-1}\frac{1}{3}$ *to the horizon.*



Sol. Let AB be the rod with a hinge at A. Let a string of tension W be attached to the rod at B and to a point C in the same horizontal plane as A.

$$\because \ BC = AC,$$

$$\angle BAC = \angle ABC = \theta.$$

The forces acting on the rod are, the reaction of the hinge at A (since the inclination of the reaction is unknown, we always take X and Y to be the horizontal and vertical components of the reaction at A), the weight W acting vertically downwards and passing through the middle point, and the tension T=W of the string.

Equating horizontal and vertical components,

$$X = T\cos(180^\circ - 2\theta) = -W\cos 2\theta, \qquad \ldots(i)$$
$$Y + T\sin(180^\circ - 2\theta) = W,$$
$$i.\,e. \quad Y + W\sin 2\theta = W. \qquad \ldots(ii)$$

Take moments about A.

$$W.\,a\cos\theta = W \times 2a\sin\theta \qquad (\because \; T = W)$$
$$\therefore \tan\theta = \tfrac{1}{2} \qquad \therefore \theta = \tan^{-1}\tfrac{1}{2}.$$

From (*i*) and (*ii*)

$$X = -W.\ \cos 2\theta = -\frac{W(1-\tan^2\theta)}{1+\tan^2\theta}$$
$$= -\frac{W(1-\frac{1}{4})}{1+\frac{1}{4}} = -W.\ \tfrac{3}{5}.$$

This shows that X acts in the opposite direction, and the figure is to be corrected.

$$Y = W - W.\ \sin 2\theta = W\left(1 - \frac{2\tan\theta}{1+\tan^2\theta}\right)$$
$$= W\left(1 - \frac{2\times\frac{1}{2}}{1+\frac{1}{4}}\right)$$
$$= W\ (1 - \tfrac{4}{5}) = \frac{W}{5}.$$

∴ Resultant reaction at A

$$= \sqrt{X^2+Y^2}$$
$$= \frac{\sqrt{10}\ W}{5}.$$

Inclination of the reaction at the hinge to the horizontal $= \tan^{-1}\dfrac{Y}{X} = \tan^{-1} - \tfrac{1}{3}$.

Exercises X

1. The following forces act along the sides of a regular hexagon ABCDEF : along BA, 7 lbs. ; BC, 5 lbs. ; DC, 1 lb.; ED, 1 lb.; FE, 1 lb. ; FA, 5 lbs. Show this system is in equilibrium.

2. Four forces represented in magnitude and lines of action by the lines AB, BC, CD, DA are in equilibrium ; show that AC and BD are parallel.

3. ABCD is a quadrilateral ; forces act along the sides AB, BC, CD, DA, measured by α, β, γ, δ times these sides respectively. Find the actual direction of the forces if there is equilibrium, and prove that $\alpha.\ \gamma = \beta.\ \delta$.

4. A circular table of weight 20 lbs. rests on three legs placed symmetrically round its edge; find the least weight which placed on the table would just cause it to overturn, and find the position of the weight.

5. A circular table of weight 40 lbs. rests on four legs placed symmetrically round its edges, find the least weight which placed on the table would just cause it to overturn.

6. A square four legged table has lost one leg; where on the table should a weight, equal to the weight of the table, be placed, so that the pressure on the three remaining legs of the table may be equal?

7. Two small rings, whose weights may be neglected, slide on the arc of a smooth vertical circle; a string passes through both rings and has three equal heavy particles attached to it, one at each end, and one between the rings; show that in the position of equilibrium, the distance between the rings is equal to the radius of the circle.

8. Prove that any coplanar system of forces, not in equilibrium, can be reduced to two forces acting at two given points A and B, one of which may be inclined at any required angle to the line AB.

9. A ladder AB rests against a smooth vertical wall at A, and is supported by a socket in which its lower end B is placed; the vertical through G (the centre of gravity of the ladder and its load) meets the horizontal through A in K, and the same horizontal cuts the vertical through B in L. Prove that the triangle BKL will serve as a triangle of forces for the weight and the reactions at A and B.

10. A smooth uniform ladder of weight 100 lbs. and length 30 ft. rests with its extremities against a smooth vertical wall at A, making an angle of 15° with the vertical, and the smooth horizontal plane at B. It is kept from slipping by a cord CD one end of which is attached to a ring on the ladder one third of the way up. A man of weight 150 lbs. climbs up the ladder to E where BE=20 ft. Find the tension in the cord and the reaction at A, and verify that the reaction at B is 400 lbs.

11. A square rests with its plane perpendicular to a smooth wall, one corner being attached to a point in the wall by a string whose length is equal to a side of the square. Show that the distances of three of its angular points from the wall are as 1, 3, and 4. *(P.U. 1931 B.A. Hons.)*

12. A rectangle ABCD in which AB, BC are respectively 8 and 12 cms., is formed of smoothly jointed rods, and is made stiff by a diagonal BD; the frame is hung up from A and a weight of 10 kilogrammes is fastened at C. Find the stresses, distinguishing between thrusts and tensions.

13. Prove that any co-planar system of forces, not in equilibrium, can be balanced by forces acting along the sides of a given triangle.

Given the system and the triangle, prove that only one set of forces will balance it.

Find the condition that the required force along any side may be zero.

14. Four light rods AB, CD, BC, AD of lengths 4, 10, 5, 5 ft. respectively, freely jointed at A, B, C, D, form a trapezium in a vertical plane, AB and CD being horizontal and AB above CD. The whole is placed on smooth horizontal supports at C and D, and a weight of 200 lbs. is placed at the middle point E of AB. Find the stresses in the rods AD, BC, CD.

15. Two equal beams AB, AC connected by a hinge at A are placed in a vertical plane with their extremities B, C resting on a horizontal plane; they are kept from falling by strings connecting B and C with the middle points of the opposite beams. Show that the tension of each string is $\frac{W}{8}\sqrt{9\cot^2\theta+1}$, θ and W being, respectively, the inclination to the horizon and the weight of each beam.

16. A triangle formed by three smooth rods is fixed horizontally and a homogeneous sphere rests on it. Prove that the pressure on each rod is proportional to its length.

17. A sphere rests on three smooth pegs which lie in a horizontal plane and are at distances a, b, c from one another. Prove that the pressures on the pegs are in the ratios

$$a\cos A : b\cos B : c\cos C.$$

18. A uniform circular disc of weight nW has a particle of weight W, attached to a point on its rim. If the disc be suspended from a point A on its rim, B is the lowest point; also, if suspended from B, A is the lowest point. Show that the angle subtended by AB at the centre of the disc is $2\sec^{-1}2(n+1)$. *(P.U. 1932)*

19. *If the algebraic sum of the moments of a system of co-planar forces is the same and of the same sign, when taken about each of three non-collinear points, prove that the system reduces to a couple.*

[**Hint :** There are *three alternatives* for the system of forces ; namely, (*i*) equilibrium ; (*ii*) reduction to a single force, (*iii*) reduction to a couple. The system is not in equilibrium as the moments have values other than zero. It cannot reduce to a single force for if F be the force we must have $Fp_1=Fp_2=Fp_3$ in *magnitude and sign.*

$\therefore$ the perpendiculars from the given points A, B, C are equal,

i. e. $p_1=p_2=p_3$.

showing that A, B, C are at equal distances on the same side of F, which is contrary to hypothesis (that A,B,C are non-collinear points). We are, therefore, left with the third alternative.]

20. Three equal light rods are freely jointed together to form an equilateral triangle ABC which is suspended from A and carries two equal weights suspended from the joints B and C, respectively. Determine the reactions at the joints.

* * * *

21. A uniform bar of length a rests suspended by two strings of lengths l and l' fastened to the ends of the bar and to two fixed points in the same horizontal line at a distance c apart. If the directions of these strings being produced meet at right angles, prove that the ratio of their tensions is

$$al+cl' : al'+cl.$$

22. A uniform sphere of weight W and radius r, rests on a smooth inclined plane and is supported by a string of length l, attached to the sphere, the other end of the string being attached to a point in the plane. Show that the string makes the angle $\sin^{-1}\dfrac{r}{r+l}$ with the plane. Determine the tension and the reaction.

23. Six equal uniform bars jointed at their extremities form a hexagon ABCDEF. The bar AF is fixed in a horizontal position and its middle point is connected by means of a string

with the middle points of the lowest bar in such a manner that the bars form a regular hexagon. Find the tension T of the string and the stresses R and S at B and C, respectively.

24. Four light rods smoothly jointed together form a parallelogram ABCD; strings connect A with C and B with D. Show that their tensions are in the ratio of their lengths.

25. Three equal uniform rods, each of weight W are hinged together to form an equilateral triangle. If the system be suspended by the middle point of one of the rods, show that the stress at the lowest angle is $\frac{1}{2\sqrt{3}}$ W, and that at each of the upper angles $\frac{\sqrt{13}}{2\sqrt{3}}$ W.

26. A regular hexagon ABCDEF is formed by six equal uniform jointed rods and is kept in shape by two light wires joining B to F and C to E. Find the force along these wires when the hexagon is suspended from A.

27. A balance has two pans, each of weight 8 ozs., suspended from a beam, each at a distance 7 in. from the pivot. A dishonest tradesman moves one pan half an inch nearer to the first, adding weight to this same pan in order that the two pans may still balance. Find how much weight he must add and by how much his profits will be increased by his dishonesty.

28. A light rod rests wholly within a smooth hemispherical bowl of radius r, and a weight W is clamped on to the rod at a point whose distances from the ends are a and b. Show that θ, the inclination of the rod to the horizon, in the position of equilibrium, is given by the equation

$$2\sqrt{r^2-ab}\sin\theta=a-b.$$

29. A uniform rectangular gate of weight W, 6 feet wide and 5 feet high, is supported by two hinges placed 6 inches from the top and bottom of the gate, respectively. If all the weight of the gate is supported by the bottom hinge, find the magnitude and directions of the reactions at the two hinges.

(D. U. 1944.)

30. One end of a uniform rod is attached to a hinge and the other end is supported by a string attached to the extremity of the rod and the rod and the string are inclined at the

same angle θ to the horizon. If W be the weight of the rod, find the tension of the string and the action at the hinge.

(P. U. 1943)

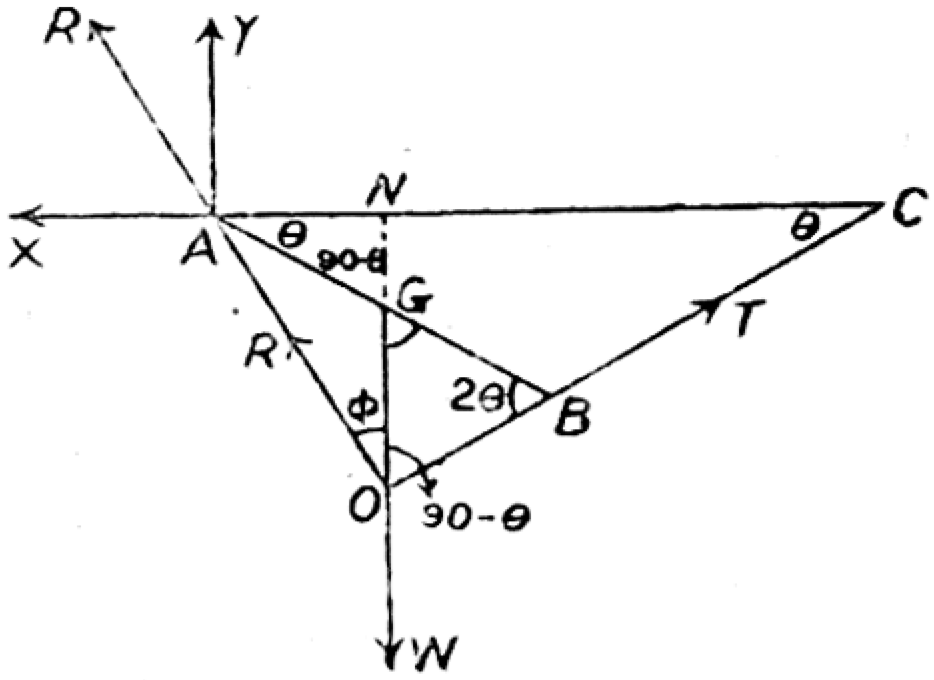
[**Hint.** We have

$$\frac{R}{\sin(90^\circ-\theta)}=\frac{W}{\sin(90^\circ-\theta+\phi)}$$

$$=\frac{T}{\sin\phi}$$

Whence, $$R=W.\frac{\cos\theta}{\cos(\theta-\phi)},$$

$$T=W\frac{\sin\phi}{\cos(\theta-\phi)} \quad \dots (1)$$



Also, from $\triangle$AOB, since AG : GB = 1 : 1

$$(1+1)\cot(90-\theta)=\cot\phi-\cot(90-\theta)$$

i.e. $$\cot\phi=3\tan\theta. \quad \dots (2)$$

$$\therefore \quad \cos\phi=\frac{3\tan\theta}{\sqrt{1+9\tan^2\theta}}, \quad \sin\phi=\frac{1}{\sqrt{1+9\tan^2\theta}} \quad \dots (3)$$

Substituting in (1) from (3), we get

$$R=\frac{W}{4}\sqrt{9+\cot^2\theta}; \quad T=\frac{W}{4}\operatorname{cosec}\theta.$$

Note.—It will be an useful exercise for the student to solve this example by resolving the forces horizontally and vertically and taking moments about A.

Revision Paper II

1. If two equal and opposite forces S in any two parallel directions, distant l apart, are combined with two like forces P, Q, in their plane and parallel to them, show that the Resultant of the forces is displaced a distance $\frac{lS}{P+Q}$

(*P. U. 1935 Supp.*)

2. ABCD is a framework of four rods in the form of a rhombus of sides l and angle α. A square CEFD of rods is structed on CD, so as to lie on that side of CD which is opposite to AB. In each of the rods AB, FE, DA, BC, DF, EC acts a force P, and in the rod CD acts a force 2P, the directions of the forces being indicated by the orders of the letters A→B in A B etc. Show that the system reduces to a single couple of moment $2lP(1-\sin\alpha)$. (*P. U. 1940.*)

3. ABCDEF is a regular hexagon and O is its centre. Forces of magnitudes 1, 2, 3, 4, 5, 6 act in the lines AB, CB, CD, ED, EF, AF in the senses indicated by the order of the letters. Reduce the system to a force at O and a couple and find the point in AB through which the single resultant force acts. *(Roorkee C. E. 1941.)*

4. Two uniform rods AB, BC, rigidly jointed at B so that ABC is a rt. angle, hang freely in equilibrium from a fixed point at A. The lengths of rods a and b and their weights are wa and wb. Prove that if AB makes an angle θ with vertical,

$$\tan\ \theta = \frac{b^3}{a^2+2ab}.$$ *(I. M. A. 1938).*

5. A bowl is formed from a hollow sphere of radius a; it is so fixed that the radius of the sphere drawn to each point in the rim makes an angle α with the vertical, and the radius drawn to a point A of the bowl makes an angle β with the vertical. If a smooth uniform rod remains at rest when placed with one extremity at A, with a point in its length on the rim of the bowl, show that the length of the rod is $4a \sin \beta \sec \frac{1}{2} (\alpha-\beta)$.

6. A rod of length $2a$ rests on a smooth vertical circle of radius b, one end being attached to a string, to the other end of which is tied a weight which hangs down over the circle. The distance of the point of contact from this end of the rod na. Prove that in the position of equilibrium the rod makes with the vertical an angle $\tan^{-1} \dfrac{2abn^2}{(1-n)(b^2-a^2n^2)}$. *(I. M. A. 1934)*

7. If forces A, B, C, D acting along the sides a, b, c, and d of a quadrilateral are in equilibrium, prove that $\dfrac{\text{AC}}{ac} = \dfrac{\text{AD}}{bd}$.

(D.E 1943.)

[**Hint.** Take moments about two opposite angular points of the quadrilateral. [Cf. Q. 10. Examples III].

8. A square table, of weight 20 lbs. has legs at the middle points of its sides, and three equal weights, each equal to the weight of the table, are placed at three of the angular points. What is the greatest weight that can be placed at the fourth corner to preserve equilibrium? *(P. U. 1933 Supp).*

9. A circular tray of radius a stands on a single circular foot of radius b. If P is the whole weight of the tray and its support, find how far from the centre a weight Q can be placed without the tray falling over. *(P. U. 1938)*

10. A square of side $2a$ is placed with its plane vertical between two smooth pegs, which are in the same horizontal line, and at a distance c Show that it will be in equilibrium when the inclination of one of its edges to the horizon is either 45° or $\frac{1}{2}\sin^{-1}\dfrac{a^2-c^2}{c^2}$. *(P. U. 1940 Supp.)*

11. A heavy uniform rod, 15 inches long, is suspended from a fixed point by strings fastened to its ends, their lengths being 9 and 12 inches. If θ be the angle at which the rod is inclined to the vertical, show that $25 \sin \theta=24$.

(P. U. 1937 Supp.)

12. A triangular lamina ABC is suspended from a point O by light strings fastened to the points A, B and hangs so that the side BC is vertical. Prove that if α, β, are the angles which the strings AO, BO make with the vertical, then $2 \cot \alpha - \cot \beta = 3 \cot B$. *(P. U. 1936)*

13. A uniform rod rests with its ends on two smooth inclined planes of inclinations α and β. Find the angle it makes with the vertical. *(P. U. 1941)*

14. One end of a string of length l is attached to one end of a smooth uniform rod of length a, and the other end of the string is attached to a light ring which slides on the rod. If the string passes over a small smooth pulley, prove that in the position of equilibrium the inclination θ of the rod to the vertical is given by $a \sin^3\theta = l \cos \theta$.

15. A uniform rod AB is in equilibrium at an angle α with the horizontal with its upper end A resting against a smooth peg and its lower end B attached to a light cord which is fastened to a point C in the same level as A. Prove that the angle β at which the cord is inclined to the horizontal is given by the equation

$$\tan \beta = 2 \tan \alpha + \cot \alpha.$$

16. Apply exercise 19 Examples X to prove Arts. 62 and 63.

17. A heavy carriage wheel of weight W and radius r, is to be dragged over an obstacle of height h, by a horizontal force F, applied to the centre of the wheel; show that F must be slightly greater than $W \frac{\sqrt{2rh-h^2}}{r-h}$.

18. A rigid wire without weight, in the form of the arc of a circle subtending an angle α at its centre, and having two weights P and Q at its extremities, rests with its convexity downwards upon a horizontal plane. Show that if θ be the inclination to the vertical of the radius to the end at which P is suspended, then

$$\tan\theta = \frac{Q\sin\alpha}{P+Q\cos\alpha}.$$

19. A beam of weight W, is divided by its centre of gravity C into two portions AC and BC, whose lengths are a and b respectively. The beam rests in a vertical plane on a smooth floor AD and against a smooth vertical wall DB. A string is attached to a hook at D and to the beam at a point P. If T be the tension of the string and θ and ϕ be the inclinations of the beam and the string respectively to the horizon, show that

$$T = W.\ a\cos\theta \,/\, (a+b)\sin(\theta-\phi).$$

20. A uniform rod of length a hangs against a smooth vertical wall being supported by a string of length l tied to one end of the rod, the other end of the string being attached to a point in the wall, prove that the rod can rest inclined to the vertical at an angle θ where $\cos^2\theta = \frac{l^3-a^2}{3a^2}$. What are the limits of the ratio $a : l$ that equilibrium may be possible?

21. Equal weights P and P attached to two strings ACP and BCP passing over a smooth peg C. AB is a heavy beam of weight W whose centre of gravity is a feet from A and b feet from B; show that AB is inclined to the horizon at at an angle

$$\tan^{-1}\left[\frac{a-b}{a+b}\tan\ \sin^{-1}\frac{W}{2P}\right]$$

22. A triangular plate ABC obtuse angled at C, stands on a table in a vertical plane having the side AC in contact with the table; show that the least weight which suspended from B will overturn it is

$$\tfrac{1}{3}\,W\,\frac{a^2+3b^2-c^2}{c^2-a^2-b^2}.$$

23. A uniform rod whose weight is W, is supported by two fine strings, one attached to each end, which, after passing over small smooth fixed pulleys, carry weights W_1 and W_2 respectively at the other ends. Show that the rod is inclined to the horizon at an angle

$$\text{Sin}^{-1}\frac{W_1^2-W_2^2}{W\sqrt{2(W_1^2+W_2^2)-W^2}}$$

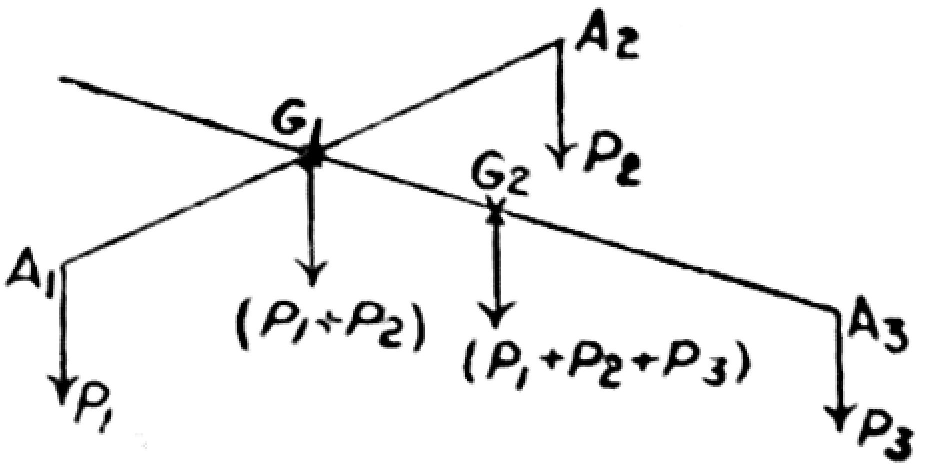
24. A smooth hemispherical bowl of diameter a, is placed so that its edge touches smooth vertical wall; a heavy uniform rod is in equilibrium, inclined at 60° to the horizon, with one end resting on the inner surface of the bowl, and the other end resting against the wall. Show that the length of the rod must be $a+\dfrac{a}{\sqrt{13}}$

25. A uniform rod of weight W, is supported in equilibrium by a string of length $2l$, attached to its end and passing over a smooth peg. If a weight W′ is now attached to one end of the rod, show that it can be placed in another position of equilibrium by sliding a length $\dfrac{lW'}{W+W'}$ of the string over the peg.

CHAPTER VII

CENTRE OF GRAVITY

74. To find the resultant and centre of any number of like parallel forces of given magnitudes applied at given points of a rigid body, *not necessarily in the same plane.*



Let P_1, P_2, P_3 ..P_n be the forces acting at A_1, A_2, A_3...A_n.

The two forces P_1 and P_2 have a resultant P_1+P_2 acting in the same direction as P_1 or P_2 at G_1, where

$$\frac{A_1G_1}{G_1A_2}=\frac{P_2}{P_1}.$$

Replacing the forces P_1 and P_2 by their resultant P_1+P_2 and combining this with the next force P_3, we see that the forces P_1, P_2, P_3 can be replaced by $P_1+P_2+P_3$ acting in the direction of the parallel forces at G_2, where G_2 lies in the line G_1A_3 and divides it in such a manner that

$$\frac{G_1G_2}{G_2A_3}=\frac{P_3}{P_1+P_2}.$$

Proceeding in this way till all the parallel forces are exhausted, we see that the resultant of the *entire system is equal to a single force* $(P_1+P_2+P_3\ ..+P_n=\Sigma P)$ acting in the same direction as the components P_1, P_2, etc., and this final resultant passes through a point G *which depends only upon (i) the positions of the points of application A_1, A_2, A_3...A_n, and (ii) the* **relative magnitudes** *of the forces, and not on their common direction.*

The point G is called the **centre of the system of parallel forces.**

We have made no assumption with regard to the *common direction* of the forces P_1, P_2, P_3......P_n. It, therefore, follows that the point G, through which resultant passes *will remain fixed,* no matter how the forces P_1, P_2, P_3...P_n are *turned* round the points A_1, A_2, A_3......A_n so as always to remain parallel amongst themselves.

Note 1. The distinction between *centre of a system of parallel forces* and *the point of application of the resultant* should be carefully observed. The resultant R may be supposed to be applied at *any point*, say K, in its line of action by the principle of *Transmissibility* of Forces. But if the forces P_1, P_2, P_3 .....acting at the points A_1, A_2, A_3.....are turned round in a different direction their resultant R $(=\Sigma P)$ will still pass through G. the *centre* of the system of parallel forces but it will no longer pass through K. Thus the *point of application* K of the resultant of like parallel forces acting at given points is not necessarily the centre of the parallel set of forces.

Note 2. It is clear that the centre of like parallel forces is unique, for the resultant cannot pass through two centres of forces except when the direction of the forces is parallel to the line joining the two centres.

75. Centre of Mass (C. M.) or Centre of Inertia (C. I).

If in the figure of the preceding article, we suppose a system of rigidly connected particles m_1, m_2, m_3 .....m_n placed at the points A_1, A_2, A_3......A_n, respectively, acted upon by *like parallel forces, the force on each particle being proportional to its mass*, the centre of this system of parallel forces is called the *centre of mass* (or *centre of inertia*) *of the system of particles.*

A rigid body may be regarded as a system of rigidly connected particles of indefinitely small mass and the *centre of mass* (or *centre of inertia*) *of a rigid body* is the centre of the system of parallel forces supposed to be acting on these particles, the force on each particle being proportional to its mass.

76. Centre of Gravity. On account of the attraction of the earth (known as *gravity*) every particle of matter is drawn towards the centre of the earth with a force proportional to its mass. This force is called the **weight** of the particle.

A rigid body is considered as an assemblage of material particles and these particles are all subject to the attractive forces of the earth. The directions of these forces which the earth exerts on the different particles composing the body, are not, strictly speaking, parallel. But *if the dimensions of the* *body are small in comparison with those of the earth, these forces may be considered to be parallel* and the resultant of this system of like parallel forces (*which is the sum of the weights of the component particles*) is called the **weight of the body**. The weight of the body being the resultant of a set of like parallel forces will pass through a definite point of the body. This point is called the **centre of gravity of the body** and *being the centre of a set of like parallel forces acting vertically downwards* can be determined by the construction of Art. 74 and as explained there, *it is a point fixed relatively to the body.*

Definition. *The centre of gravity of a body may, therefore, be defined to be that point fixed in the body or rigidly connected with it through which the line of action of the weight of the body always passes whatever be the position of the body provided that its size and shape remain unaltered.*

Centre of gravity is denoted by the letters C. G. Since the weights of the particles constituting a body are practically proportional to their masses when the body is outside the earth and near its surface, the *centre of mass* (centre of inertia) of a body *practically coincides with its centre of gravity.*

Note 1. The centre of gravity does not necessarily lie in the body itself. Thus the centre of gravity of a circular material ring is at its centre, which point is rigidly connected with the ring.

Note 2. The centre of gravity of a straight thin wire is at its centre. But if the wire be bent in the form of a circular arc, its centre of gravity will be outside the matter constituting the arc. *A different arrangement of particles usually alters the centre of gravity, or centre of mass.*

77. Distinction between centre of gravity (C. G.) and centre of mass (C. M.)

From what has been stated in the foregoing articles, it follows that a body has a centre of gravity only—

(*i*) when it is acted on by gravity, *i. e.*, when it has weight.

(*ii*) and secondly, when the weights of the particles constituting the body are supposed to form a system of like parallel forces, proportional to the masses of these particles.

If a body be imagined to be removed to an infinite distance in space where the earth ceases to exercise any attraction on it or if it be imagined to be removed to the centre of the earth where the force of gravity is nil, the body will have no weight ; and it is meaningless to speak of the existence of a point in the body through which its weight always acts. *Under these circumstances, the body will have no centre of gravity.* But the body will retain its mass which, being an inherent property of matter is quite independent of gravity. The body, therefore, will have a *centre of mass.*

Again, the existence of the centre of gravity of the body depends on the parallelism of the weights of its constituent particles but this *assumption* is only justified in the case of *rigid bodies whose size is small in comparison with the dimensions of the earth.*

If the volume of the body be not small in comparison with that of the earth, the weights of the particles composing the body cannot be considered parallel and it will, therefore, have no centre of gravity.

The *centre of gravity* of a body, *when it exists,* being the centre of a particular set of parallel forces, namely, the weights of its constituent particles, must *be the same as its centre of mass.* (Art. 76).

We, therefore, **summarise our conclusions** as follows :—

1. A rigid body small in size as compared with the earth and situated on or near its surface, has a centre of gravity.

2. A very large body, that is, a body which cannot be considered small, in comparison with the earth has not necessarily a centre of gravity but it has a centre of mass.

A body spherical in shape however large it may be ,has a centre of gravity.

3. A body imagined to be removed to an infinite distance in space where earth's attractive force is inoperative or a body imagined to be removed to the centre of the earth where gravity is nil, has no centre of gravity.

4. The centre of gravity of a body if it exists, coincides with its centre of mass.

Note. Every rigid body dealt with in this book will be assumed to have a centre of gravity.

78. *The effect of gravity on any body is the same as if the whole mass of the body were collected at the centre of gravity.* This follows from the construction of Art. 74 which is also the construction for the centre of gravity Art. 76.

The C. G. of a system of rigid bodies, when the C. G. of each body is known, can be easily deduced from the above. For the weights of the component bodies from a set of parallel forces acting at their centres of gravity and the centre of these parallel forces is the C. G. of the system.

79. *If a body acted upon by gravity only be supported at its centre of gravity, it will rest in any position.*

For, in every position, the weight of the body acts vertically downwards through the point of support and will, therefore be balanced by its equal and opposite reaction.

Note that the point of support is fixed, and that the weight of the body and the reaction of the support on the body pass through it. These cannot turn the body round, for they have no moment about that point.

80. A body cannot have more than one centre of gravity.

For if possible, *assume* that there are two centres of gravity G and G′. Let the body be so placed as to have G G′ *horizontal*, then by definition, the line of action of the weight of the body must pass through G and G′; but this is impossible since it contradicts the well-known fact that this line is always vertical.

Hence there can be only one centre of gravity.

The position of the C. G. of a lamina may be experimentally found thus :—

Let a body be suspended first from a point O and then from a point O′. Assuming the connections at O and O′ to be frictionless, we see that the only forces acting on the body in either case are its weight and the reaction at the point of support. Hence from equilibrium, OG and O′G must be vertical. These vertical lines through O and O′ may be marked by means of a plumb-line. G is then their point of intersection.

80. Position of C. G. found by inspection. The centre of gravity of a body which is symmetrical with respect to a point will coincide with the point ; * for the body may be supposed to be composed of pairs of particles of equal weight lying on lines passing through the centre of symmetry and at equal distances from it. Similar remarks apply to the position of C. G. in a body which is symmetrical with respect to a straight line or a plane. Thus

(*a*) *the centre of gravity of a uniform straight rod is at its middle point,* for it is the point of symmetry for the rod ;

(*b*) *the centre of gravity of a circle, a circular ring, a sphere, or a spherical surface is its centre* ;

(*c*) *the centre of gravity of a parallelogram, or a parallelopiped is the intersection of its diagonals* ;

(*d*) *the centre of gravity of a triangular lamina is the centroid of the lamina,* for a median of a triangle is a line of summetry for the triangle.

(*e*) *the centre of gravity of a right circular cylinder lies at the middle point of its axis,* for its axis is a line of symmetry and the plane bisecting the axis perpendicularly is the plane of symmetry ;

(*f*) *the centre of gravity of a right circular cone lies on its axis.*

---

*Two points P and Q are said to be symmetrically placed with respect to a point O, or a straight line l, or a plane α, when PQ is bisected at O, or by l at right angles, or by α perpendicularly.

A figure is said to be symmetrical with respect to a given point, or a straight line, or a plane, when there exist such pairs of points. For instance, a circle is symmetrical with respect (*i*) to its centre, (*ii*) to any diameter, and (*iii*) to the normal to its plane through the centre, a sphere is symmetrical with respect (*i*) to its centre, (*ii*) to any diametre, and (*iii*) to any plane through the centre ; and a cube is symmetrical with respect (*i*) to its centre, and (*ii*) to a plane through the centre parallel to any two opposite faces, and has evidently three axes of symmetry.

Determination of the C. G. of a few elementary bodies.

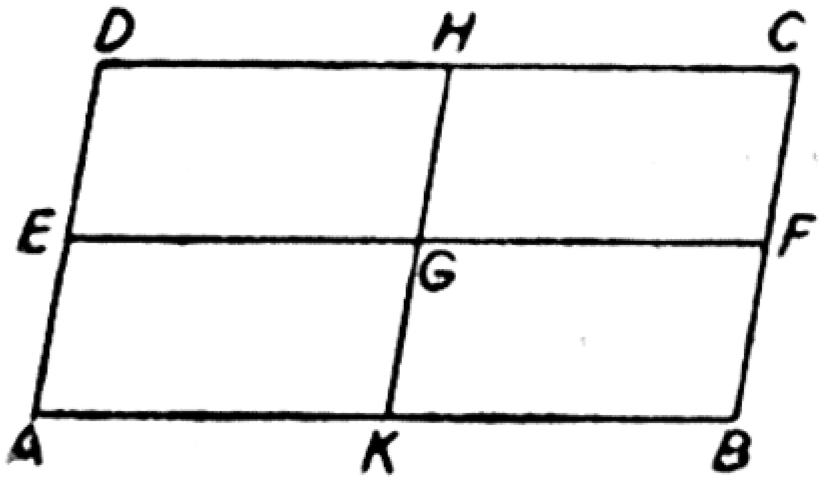
82. *To find the C. G. of a material straight line.*

Let AB be the given straight line.

We may suppose the line to be made-up of equal particles equidistant from its middle point C. The C. G. of each pair of particles (such as P and Q) will be at the middle point C of the line (PQ) joining them. Therefore C will be the centre of gravity of the whole line AB.

82. 1. *To find the C. G. of a lamina in the form of a parallelogram.*

Let ABCD be a parallelogram, regarded as a *lamina or a thin sheet of matter.*



The parallelogram may be supposed to be made up of an indefinitely great number of material straight lines each parallel to AB.

The C. G. of each is at its middle point.

The line HK which joints the middle points of AB and CD will bisect every such line parallel to AB.

∴ C. G. of the parallelogram is in HK.

Similarly the C. G. is in EF which joins the middle points of BC, AD.

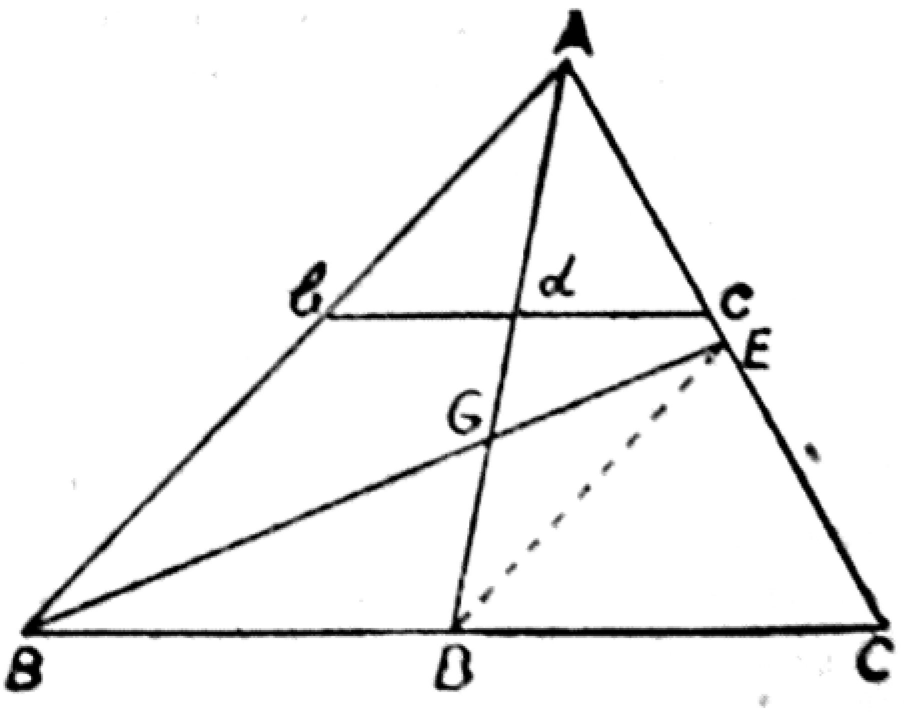
∴ G, the point of intersection of EF and HK is the C. G. of the parallelogram.

83. *To find the C. G. of a lamina in the form of a plane triangle.*

The triangle ABC may be supposed to be made up of an indefinitely great number of material straight lines parallel to the side BC.

Let *bc* be one of these lines.

Join A to D, the middle point of BC.



Now AD bisects every line such as *bc* parallel to BC.

∴ C. G. of the △ is in AD, the *median to the side BC.* Similarly, C. G. of the △ is in BE, the *median to the side CA.*

∴ G, the *point of intersection of the medians* is the C. G. of the △.

83. 1. *To determine the position of G in the median AD.*

Now, $DE=\frac{1}{2}BA$, and ∥ to BA

[∵ E, D are mid-points of CA and CB.

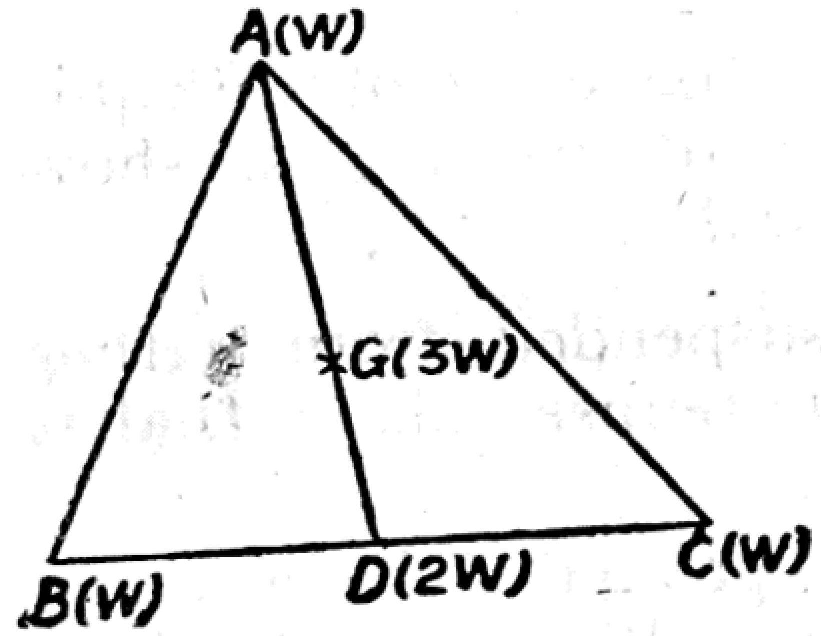
Again, △s DGE, AGB are equiangular, ∴ similar,

$$\text{Hence } \frac{DG}{DE}=\frac{GA}{AB}=\frac{GA}{2DE}$$

$$\therefore\ DG=\tfrac{1}{2}GA. \qquad \therefore\ DG=\tfrac{1}{3}AD.$$

Thus, *the C. G. of a triangle lies in any median at one-third of its length from the corresponding side,* or *the C. G. of a triangle is the point of intersection of any two medians.*

84. *To show that the C. G. of a triangle coincides with the C. G. of three particles of* **equal masses** *placed at its angular points.*



Let ABC be the triangle. Bisect BC at D; then D is the C. G. of equal particles at B and C.

W being the weight of each particle, we have now 2W at D and W at A.

Their C. G. is at the point O in AD such that

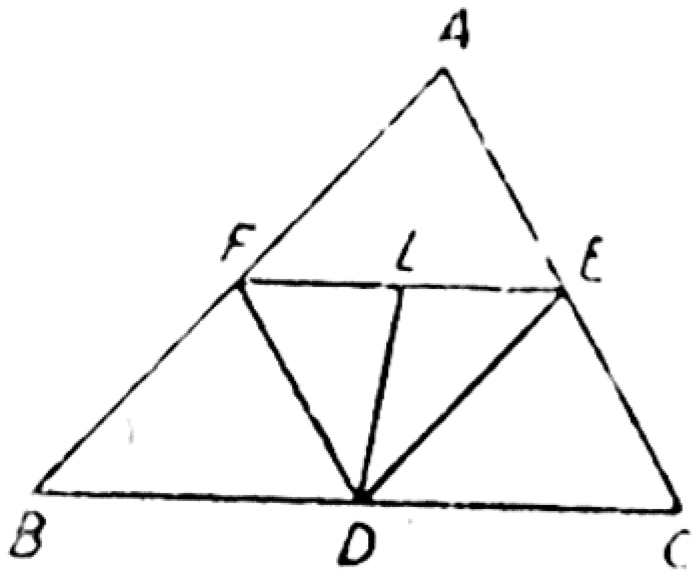
$$\frac{AO}{OD}=\frac{2W}{W}=2$$

$$\therefore\ DO=\tfrac{1}{2}AO \qquad \therefore\ DO=\tfrac{1}{3}AD.$$

∴ O coincides with G, the C. G of the triangle.

Ex. Show that the C. G. of a triangle is the same as the C.G. of three equal particles placed at the middle points of its sides.

85. *To find the C. G. of a wire bent into the form of a triangle.*



Let ABC be the form of the wire. D, E, F, the middle points of the sides BC, CA, AB are the centres of gravity of BC, CA, AB.

Evidently $EF=\frac{1}{2}BC$, $FD=\frac{1}{2}CA$, $DE=\frac{1}{2}AB$.

Masses proportional to BC, CA, AB are at D, E, F.

If L be the centre of gravity of masses at E and F.

EL : FL = mass at F : mass at E

= AB : AC = DE : DF.

∴ DL is the internal bisector of ∠EDF.

Thus the C. G. of the wire lies on the internal bisector of ∠EDF.

Similarly the C. G. lies on the internal bisector of ∠DEF.

Hence the C. G. (of the wire, or) of **the perimeter of the triangle, is at the incentre.**

Exercises XI

1. If a heavy uniform lamina, in the shape of an equilateral triangle, be suspended from any of its angles, show that the opposite side is always horizontal.

2. If a right angled triangle be suspended from either of the points of trisection of the hypotenuse, show that it will rest with one side horizontal.

3. A triangular lamina, when suspended from a point P in the side AB, rests with the side BC vertical; show that AP=2BP.

4. Show that the locus of the C. G.'s of all right-angled triangles which can be described on the same hypotenuse is a circle, and find the radius of this circle.

5. If G be the C. G. of a triangle ABC, whose sides are 3, 4, 5, prove that $3(GA^2+GB^2+GC^2)=50$.

6. If a parallelogram be divided into four triangles by its diagonals and the C. G.'s of these triangles be joined,

show that these joining lines will form a parallelogram.

7. If the C. G. of a triangle coincides with the centre of the circumscribed circle, the triangle is equilateral.

8. If the lengths of the sides of a triangular lamina are 3, 4, and 5 feet, find the distance of the C. G. from each of its sides.

9. The equal sides of an isosceles triangular lamina ABC are 10 feet in length and its base BC is 12 feet. Find the distance of its C. G from the sides.

[*Hints.* If p_1, p_2, p_3 are the perpendiculars from A, B, C on the opposite sides, then $\sqrt{s(s-a)(s-b)(s-c)}$

$$=\frac{ap_1}{2}=\frac{bp_2}{2}=\frac{cp_3}{2} \quad \text{where, } s=\frac{a+b+c}{2}$$

$$=\frac{10+10+12}{2}=16.$$

The distances of C. G. from the sides are evidently

$$\frac{p_1}{3},\ \frac{p_2}{3} \text{ and } \frac{p_3}{3}.$$

In this particular case $b=c$ and $\therefore\ p_2=p_3$.]

10. If two triangles stand on the same base, show that the line joining their centres of gravity is parallel to the line joining their vertices.

11. The base of a triangle is fixed; its vertex moves on a given straight line; show that the centre of gravity also moves on a straight line.

12. The base of a triangle is fixed, and it has a given vertical angle; show that the C. G. of the triangle moves on an arc of a certain circle.

13 The sides of a uniform triangular lamina are 5, 12 and 13 centimetres in length. If it be suspended from the vertex opposite to the shortest side, find the inclination of this side to the horizon.

14. A uniform bar of length 30 feet and of weight 15 lbs. is bent into the form of a triangle of sides 5, 12, 13 feet, respectively, and rests on a rough horizontal table. Find the least force which, applied at a corner, is on the point of raising that corner from the table.

15. Show that the C. G. of four equal weights, at the vertices of a quadrilateral coincides with the C. G. of the parallelogram formed by joining the middle points of the sides.

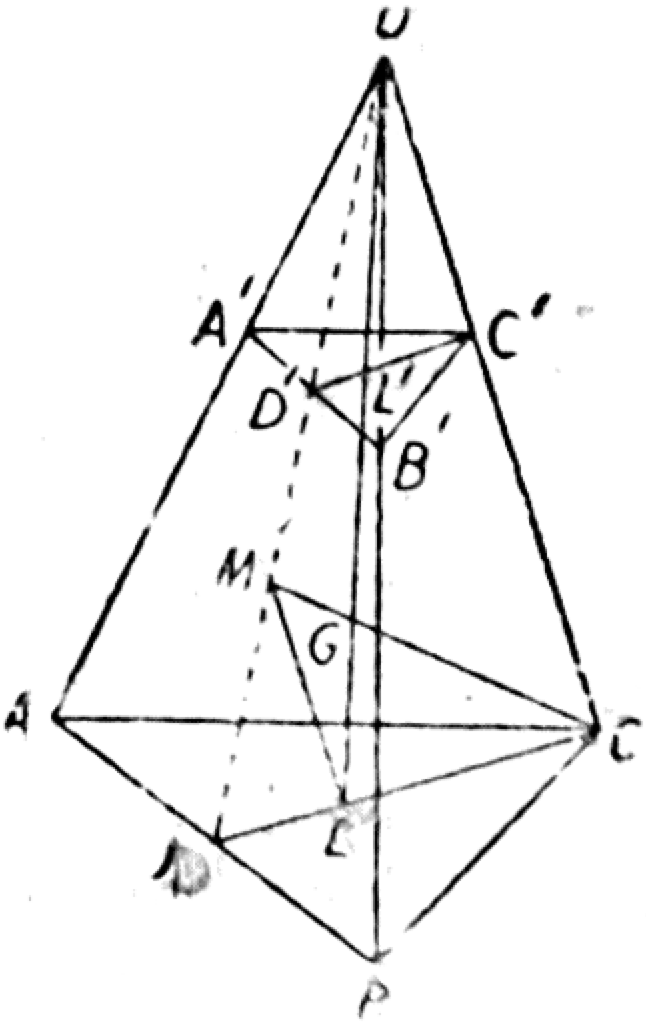
16. The C. G. of a quadrilateral lamina lies on one of its diagonals; show that this diagonal bisects the other diagonal.

17. *If the centre of gravity of a quadrilateral lamina coincides with the point of intersection of the diagonals; show that the quadrilateral is a parallelogram.*

86. *To find the centre of gravity of a uniform* **tetrahedron** *or a* **pyramid** *on a triangular base.*

Let A′B′C′ be a section of a tetrahedron OABC by a plane parallel to the base ABC, and L the centre of gravity of △ ABC. Let OL cut the plane A′B′C′ in L′. Then L′ is the C. G. of the △ A′B′C′.

Since any plane cuts parallel planes in parallel lines.



∴ A′B′ ∥ AB, B′C′ ∥ BC, C′A′ ∥ CA.

∴ △s A′B′C′ and ABC are similar.

And OD bisects A′B′ at D′, OD being the median of △ OAB.

Since C′D′ ∥ CD and OL′L meets them at L′ and L,

∴ D′L′ : L′C′ = DL : LC.

Hence L′ is the C.G. of △ A′B′C′.

Assuming the tetrahedron to be divided into triangular strips parallel to the base ABC, we see that the C.G. of the tetrahedron lies on OL.

Similarly the C.G. of the tetrahedron lies on CM where M is the C.G. of △ OAB.

Then G, the point of intersection of OL and CM, is the centre of gravity of the solid.

Since $DL = \frac{1}{3}DC$, $DM = \frac{1}{3}DO$,

∴ $LM \parallel OC$ and $LM = \frac{1}{3}OC$.

Note—*The centre of gravity of a tetrahedron coincides with the centre of gravity of four equal particles placed one at each of* *its angular points.* The centre of gravity of three equal particles each of weight W placed at A, B, C is at L, the centre of gravity of △ ABC.

∴ The centre of gravity of two particles—one of weight W at A and the other of weight 3 W at L—will divide OL at G′ such that OG′ : G′L = 3 : 1.

Thus G′ coincides with G.

When we know that the C. G. of the tetrahedron lies on both OL and CM and that of four equal particles placed at the angular points also on these lines, the actual position of the C. G. of the tetrahedron may more easily be found from the latter fact. Thus we have

$$OG = \tfrac{3}{4}OL.$$

87. *To find the centre of gravity of any pyramid having a plane polygonal base.*

Divide the given pyramid into slices parallel to the base. All such slices have for their faces polygons similar to the base. It may be shown, as in the preceding article, that the C. G. of every polygon lies on the straight line joining O, the vertex of the pyramid, and L, the centre of gravity of the base. Hence the C. G. of the pyramid lies on OL.

Now take any point in the base and join it to its angular points and the vertex of the pyramid. The pyramid will, therefore, be divided into triangular, pyramids.

The centres of gravity of these tetrahedra lie on a plane parallel to the base, at one-fourth of the distance of the common vertex from it.

Hence the centre of gravity, G of the pyramid, is the point of intersection of OL with this plane and at a distance from the base equal to one-fourth of the distance of the vertex O from the base, *i.e.*, $OG = \tfrac{3}{4}OL$.

88. *To find the centre of gravity of a* **cone.**

(*i*) **If the cone is solid,** its centre of gravity is on the line joining the vertex to the centre of gravity of the base at one-fourth of the length of this line from the base; for the cone can be split up into infinitesimal pyramids having a common vertex.

In the case of a right circular cone, the centre of the circular base is the C. G. of the base.

Hence the C. G. of a right solid cone is on the axis at one-fourth of its length from the base.

(*ii*) **Let the cone be hollow.**

Take A_1, A_2......A_n points on the boundary of the base. Join them to the vertex O. We get the slant surfaces of the pyramid whose vertex is O and base is the polygon A_1, A_2...... A_n. The centre of gravity of each triangle is at a distance from its base equal to one-third the distance of the vertex O.

Hence the C. G. of the whole surface is at a height above the base equal to one-third the height of the pyramid.

If the number of the sides of the polygonal base indefinitely increased, the slant surface of the pyramid, will ultimately become the slant surface of the cone.

Thus the C. G. of the slant surface of a cone lies in a plane parallel to the base and at a distance from the base equal to one-third of the height.

The actual position of the C. G. cannot be found by an elementary method.

But if the cone be right, the C. G. *of the slant surface is on its axis at a distance from the base equal to one-third of the height.*

Note. It may be noticed that the *surface of the base* is not taken into consideration while finding the C. G. of the surface of a pyramid or a cone.

89. Before we solve some examples on the foregoing articles, we draw the attention of the students to the *following theorems* whose application is wide and which have already been proved in a different form : the resultant of two parallel forces divides the line joining their points of application in the inverse ratio of the forces.

Theorem (*a*) **C. G. of a compound body.** *Given the centres of gravity of two parts which compose a body, to find the centre of gravity of the whole body.*

Method. (*a*) Let G_1, G_2 be the centres of gravity of the parts having weighing weights W_1, W_2.

Divide the line $G_1 G_2$ internally at G, such that

$$\frac{G_1G}{GG_2} = \frac{W_2}{W_1} \qquad \text{... (A).}$$

The G is the C. G. of the whole body.

This result may be found directly as follows :—

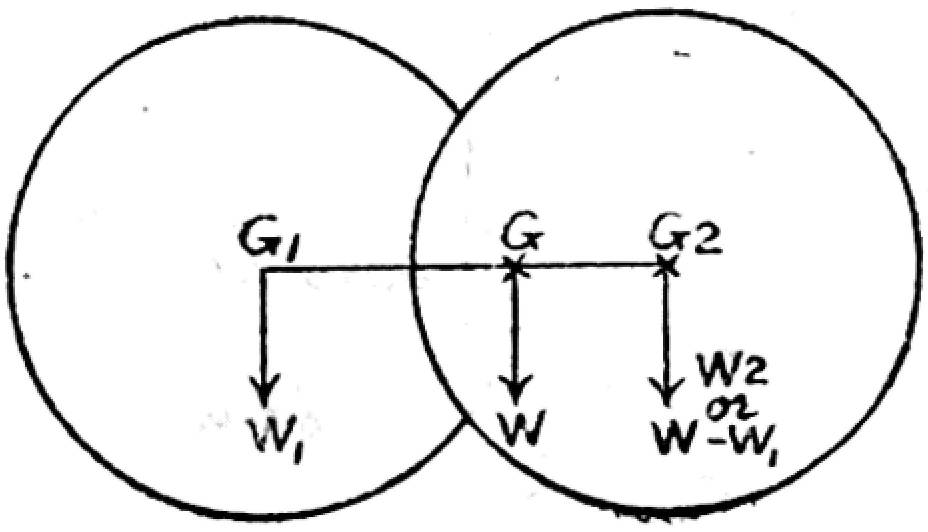
G is the C. G. of W_1 at G_1 and $W - W_1$ at G_2,

$\therefore$ $(W - W_1)\, GG_2 = W_1 . G_1G_2$.

If it is necessary to find the distance of G from G_1 or G_2 in terms of the distance G_1G_2, we have from (A),

$$\frac{G_1G}{W_2} = \frac{GG_2}{W_1} = \frac{G_1G_2}{W_1 + W_2}$$

$$\therefore\ G_1G = \frac{W_2}{W_1 + W_2}\, G_1G_2 \ \text{ and } \ GG_2 = \frac{W_1}{W_1 + W_2}\, G_1G_2.$$



(*b*) *Given the centre of gravity of a part of a body and also the centre of gravity of the whole, to find the centre of gravity of the remainder.*

Theorem. (*b*) Let G_1 be the C.G. of a part of weight W_1, and G that of the whole body having weight W.

Divide the line G_1G externally at G_2, such that

$$\frac{G_1G_2}{GG_2} = \frac{W}{W_1}.$$

Then G_2 is the C. G. *of the remainder.*

Subtracting unity from both sides we have

$$\frac{G_1G_2 - GG_2}{GG_2} = \frac{W - W_1}{W_1},$$

$$\textit{i.e.}\ \frac{G_1G}{GG_2} = \frac{W - W_1}{W_1},$$

$$\text{Hence } GG_2 = \frac{W_1}{W - W_1}\, G_1G.$$

This equation gives the position of G_2 when those of G and G_1 are known.

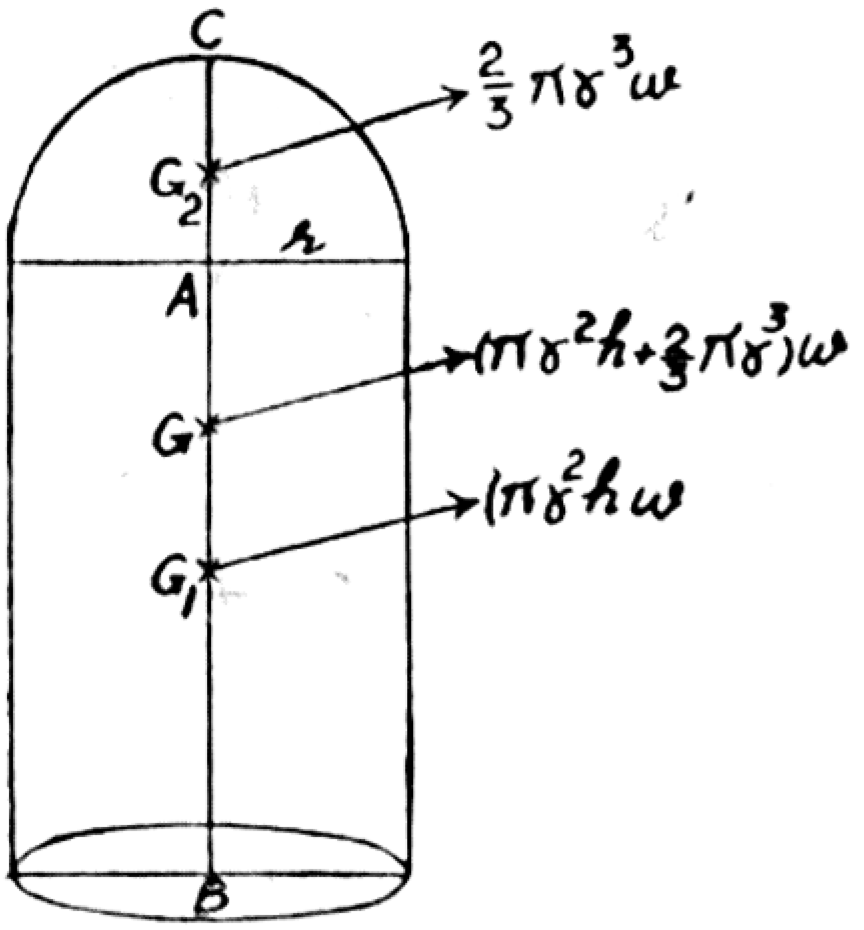
Note 1. Theorom (*b*) determines the G. G. of a body when a *portion is removed.*

If G_2, the C. G. of the removed part (*the remainder*) coincides with G, the C. G. of the whole body, then GG_2 is zero.

$\therefore$ GG_1 is also, zero *i.e.*, G_1 also coincides with G, and the centres of gravity of the two parts and the whole are all coincident.

Note. 2. By way of illustration, we may point out that the C. G. of a trapezium can be found by dividing it into triangles, or into a triangle and a parallelogram, and that the C. G. of the sector of a circle can be deduced from the segment of a circle, and vice versa.

Ex. 1. A homogeneous solid is formed of a hemisphere of radius r soldered to a right circular cylinder of the same radius. If h be the height of the cylinder, find the distance of the C. G. of the solid from the common base.



$$\left.\begin{aligned} AG_2 &= \frac{3}{8}\,r \\ AG_1 &= \frac{h}{2} \end{aligned}\right\} \quad (\textit{given})$$

$$\therefore\ G_1G_2 = \frac{3}{8}\,r + \frac{h}{2}.$$

If w be the weight of unit volume of the solid, then

Weight at G_1 is $\pi r^2 h w$

" " G_2 is $\frac{2}{3}\pi r^3 w$.

If G be the C. G. of the compound body, we have

$$GG_1 \times \pi r^2 h w = GG_2 \times \tfrac{2}{3}\ \pi r^3 w$$

$$i.e.\ GG_1 \times h \doteq GG_2 \times \tfrac{2}{3} r$$

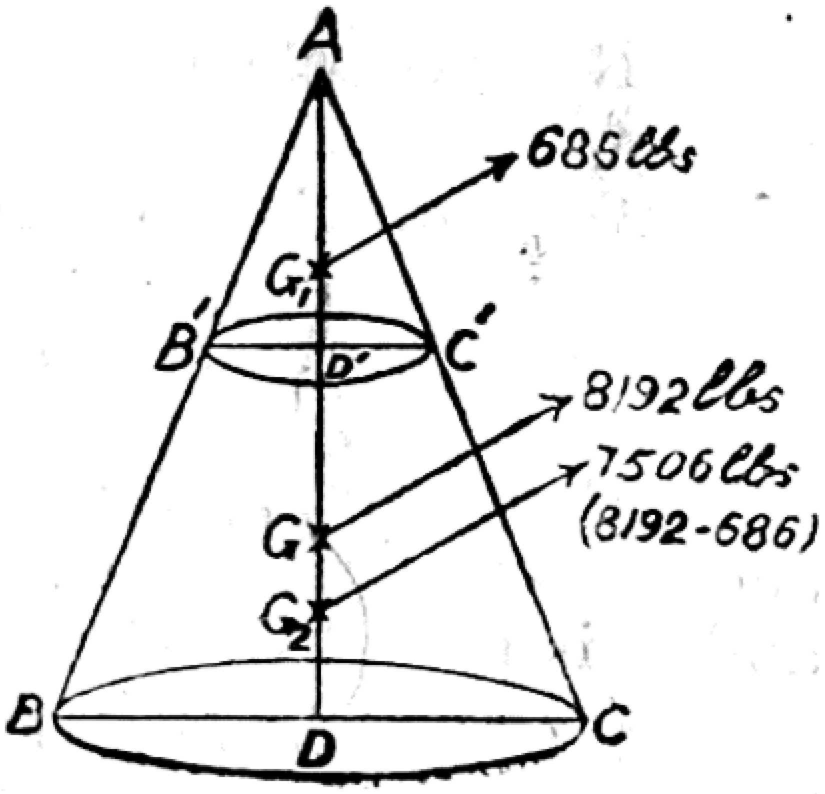
$$i.e.\ \frac{GG_1}{\frac{2}{3}r} = \frac{GG_2}{h} = \frac{G_1G_2}{\frac{2}{3}r + h} = \frac{\frac{3}{8}r + \frac{h}{2}}{\frac{2}{3}r + h}$$

$$\therefore\ AG = GG_2 - \tfrac{3}{8}r = h \times \left(\frac{\frac{3}{8}r + \frac{h}{2}}{\frac{2}{3}r + h}\right) - \tfrac{3}{8}r$$

$$= \tfrac{3}{4}\,\frac{2h^3 - r^2}{3h + 2r}.$$

Note.—G is *on* the base, *below* the base or *above* the base according as $2h^2-r^2$ is equal to, greater than, or less than zero *i.e.*, according as h is equal to, greater than or less than $\frac{r}{\sqrt{2}}$.

Ex. 2. A solid right circular cone of height 64 inches and mass 8192 lbs. is cut by a plane perpendicular to its axis so that the mass of the small cone removed is 686 lbs. Find the height of the C. G. of the truncated portion above the base of the cone. [*P. U. 1943*]



$$\frac{\text{wt. of A B}'\text{C}'}{\text{wt. of ABC}}=\frac{686}{8192}=\frac{343}{4096}\ (\textit{given})$$

$$\therefore\ \frac{\text{vol. of AB}'\text{C}'}{\text{vol. of ABC}}=\frac{343}{4096}=\left(\frac{7}{16}\right)^3$$

$$=\left(\frac{\text{AD}'}{\text{AD}}\right)^3.$$

$\therefore\ \text{AD}'=\frac{7}{16}\text{AD}=\frac{7}{16}\times 64=28$ in.

$\therefore\ \text{AG}_1=\frac{3}{4}\times 28=21$ in. ; also,
$\text{AG}=\frac{3}{4}\times 64=48$ in.

$\therefore\ \text{GG}_1=48-21=27$ in.

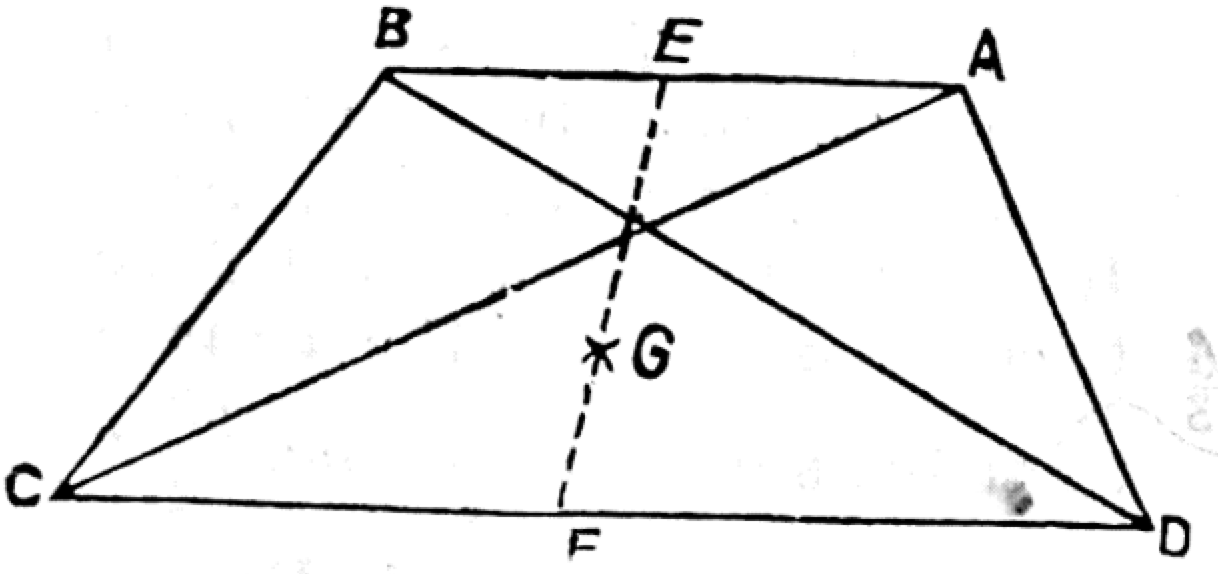
$$\therefore\ \text{GG}_1\times 686=\text{GG}_2\times 7506.$$

$$\therefore\ \text{GG}_2=\frac{686}{7506}\times 27=\frac{343}{139}\text{ in.}$$

Hence $\text{DG}_2=\text{DG}-\text{GG}_2=16-\frac{343}{139}=13\frac{74}{139}$

$=13{\cdot}532$ in. to 3 places of decimals.

90. *Find the centre of gravity of a* **trapezium.**



Let ABCD be the trapezium having AB and CD parallel and equal to $2a$ and $2b$, respectively.

Let E and F be the middle points of AB and CD.

Let p be the perpendicular distance between AB and CD. The trapezium may be divided into two triangles ABC and ACD by the diagonal AC, and the areas of these triangle are pa and pb ; therefore their weights are proportional to a and b.

Replace them by particles equal to one-third of their weights placed at their angular points.

We thus have weights proportional to

$$\frac{a}{3}+\frac{b}{3} \text{ at each of A and C,}$$

$$\frac{a}{3} \text{ at B and } \frac{b}{3} \text{ at D.}$$

Again, the diagonal BD divides the trapezium into two triangles ABD and BCD, whose weights are also proportional to a and b.

Replace them similarly by particles placed at their angular points.

We thus have weights proportional to

$$\frac{a}{3}+\frac{b}{3} \text{ at each of B and D, } \frac{a}{3} \text{ at A, } \frac{a}{3} \text{ at C.}$$

Hence the C. G. will be the same when the latter weights are added to the former, i.e., when we have *double the trapezium*, and it is the C. G. of the weights proportional to

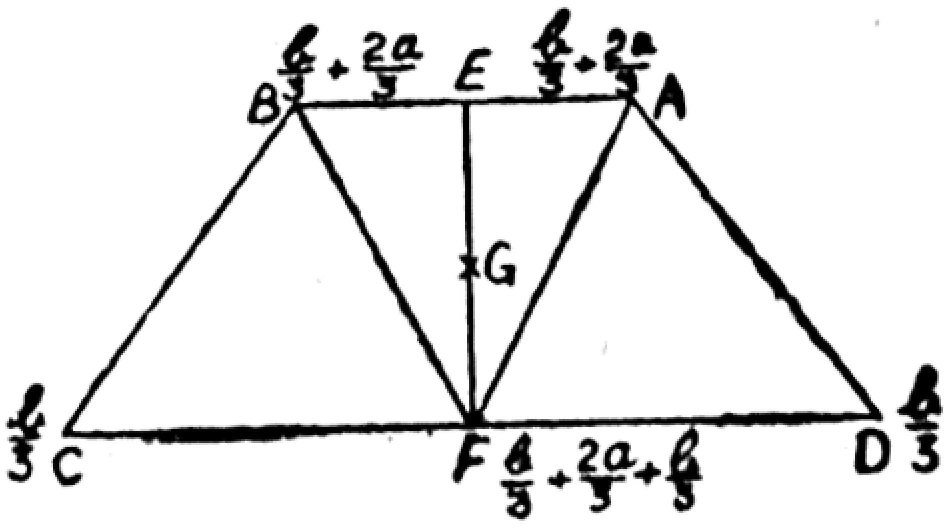
$\frac{1}{3}(2a+b)$ at A, $\frac{1}{3}(2a+b)$ at B, $\frac{1}{3}(a+2b)$ at C,

$\frac{1}{3}(a+2b)$ at D.

The resultant of the weights at A and B is proportional to $2a+b$ and acts at E, the mid-point of AB.

The resultant of the weights at C and D is proportional to $a+2b$ and acts at F, the mid-point of CD.

Hence the C. G. of the trapezium is on EF at G such that EG : GF $=2b+a : 2a+b$.



$\frac{b}{3}+\frac{2a}{3}$
E
$\frac{b}{3}+\frac{2a}{3}$
B
A
G
$\frac{b}{3}$
C
F
$\frac{b}{3}+\frac{2a}{3}+\frac{b}{3}$
D
$\frac{b}{3}$

Alternative Method.

By joining AF and BF, the trapezium may be divided into *three triangles* BCF, BFA, AFD. If p be the perpendicular between AB and CD, the areas of these triangles are $\frac{pb}{2}$, pa, $\frac{pb}{2}$.

$\therefore$ their weights are proportional to b, $2a$, b. Replace them by particles equal to *one-third* of their weights at the vertices of these triangles. We, thus, have weights *proportional to*

$$\left.\begin{array}{l}\frac{b}{3}+\frac{2a}{3}\text{ at each of A and B}\\ \frac{b}{3}\text{ at C and }\frac{b}{3}\text{ at D}\\ \frac{b}{3}+\frac{2a}{3}+\frac{b}{3}\text{ at F.}\end{array}\right\}\ \textit{i.e., weights proportional to}\ \left\{\begin{array}{l}\frac{2b}{3}+\frac{4a}{3}\text{ at E}\\ \frac{2b}{3}\text{ at F}\\ \frac{2a}{3}+\frac{2b}{3}\text{ at}\end{array}\right.$$

i.e., weights proportional to $\frac{2b}{3}+\frac{4a}{3}$ at E and $\left(\frac{2a}{3}+\frac{2b}{3}\right)+\frac{2b}{3}$ at F.

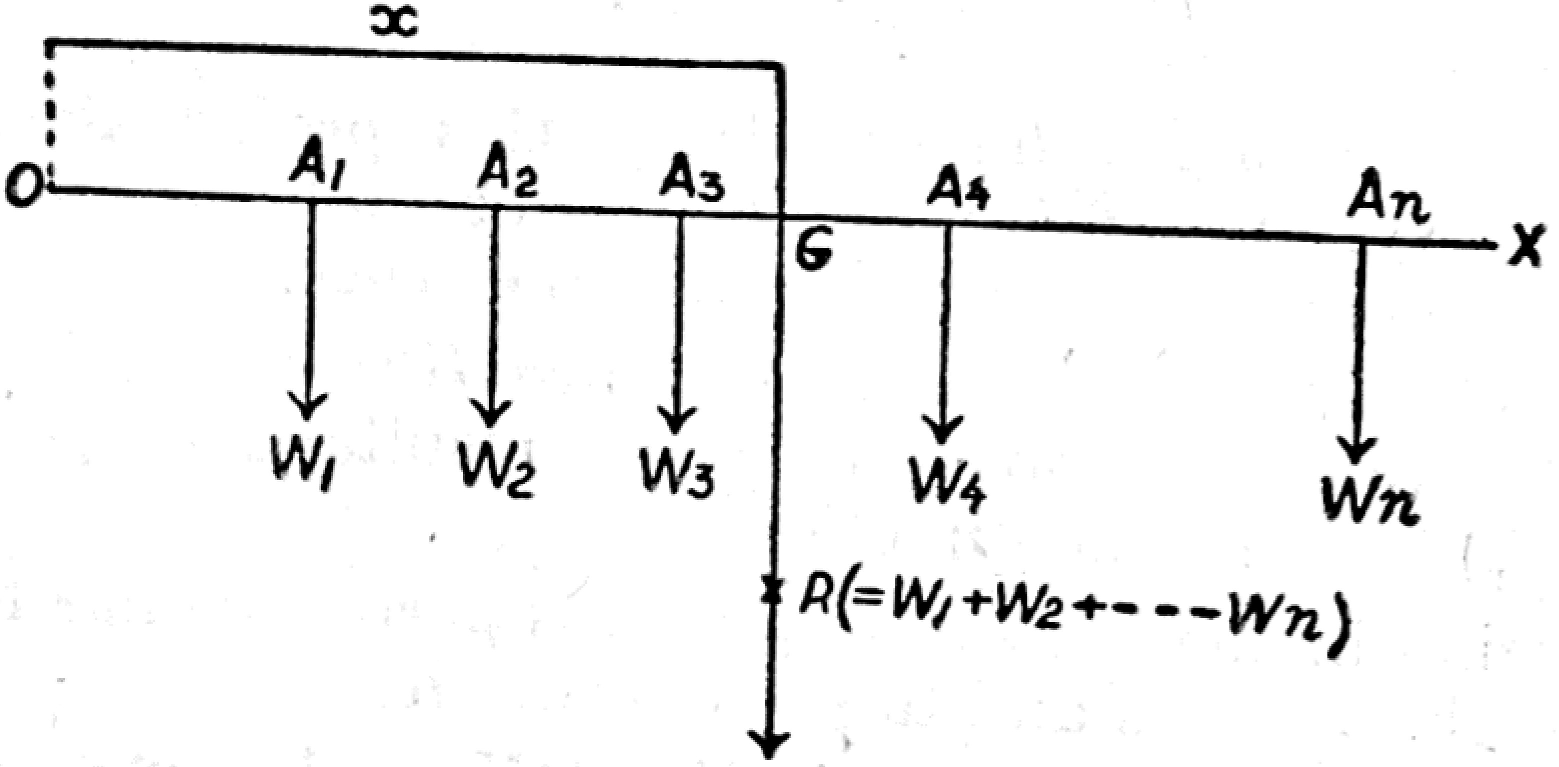
The problem, therefore, reduces to finding the C. G. of weights proportional to $\frac{2b}{3}+\frac{4a}{3}$ at E and $\frac{2a}{3}+\frac{4b}{3}$ at F.

Hence, G, the required C. G. lies on EF such that

$$\left(\frac{2b}{3}+\frac{4a}{3}\right)\text{EG}=\left(\frac{2a}{3}+\frac{4b}{3}\right)\text{GF}.$$

Hence the C. G. of the trapezium is at G on EF such that $\frac{\text{EG}}{\text{GF}}=\frac{2b+a}{2a+b}$.

Q.P. **91. C. G. of weights in the same straight line.** *To find the centre of gravity of a series of heavy particles of weight $W_1, W_2, W_3, \ldots W_n$,* **placed at given points in a straight line.**



Let $A_1, A_2, A_3 \ldots A_n$ be the points on the line OX, which for convenience we shall consider the horizontal distances of the points from a fixed point O in OX, being $x_1, x_2, x_3, \ldots x_n$ respectively. At $A_1, A_2, A_3 \ldots A_n$ are placed particles of weights $W_1, W_2, W_3 \ldots W_n$.

Let the resultant R $(=W_1+W_2+W_3 \ldots W_n)$ of the system of parallel forces W_1, W_2, $W_3 \ldots W_n$ act vertically downwards at G, where $OG=x$. x is to be in terms of x_1, x_2, x_n, and W_1, $W_2 \ldots W_n$. We have, by taking moments about O, of the component forces and the resultant,

$$OG \times (W_1+W_2+W_3+\ldots+W_n)$$
$$=W_1x_1+W_2x_2+W_3x_3+\ldots W_nx_n.$$

$$\therefore \quad \bar{x}=OG=\frac{W_1x_1+W_2x_3+W_3x_3+\ldots+W_nx_n}{W_1+W_2+W_3+\ldots W_n}.$$

Note. This formula will remain true even if the directions of W_1, W_2 etc. are at right angles to OX.

92. C. G. obtained by taking moments about a line. *To find the centre of gravity of a series of heavy particles of weights W_1, W_2, $W_3 \ldots\ldots W_n$* **placed at given points lying in a plane.**

Let A_1, A_2 $A_3 \ldots\ldots A_n$ be the points in a plane and let the co-ordinates of these points referred to two rectangular axes OX, OY, in the plane be (x_1, y_1), (x_2, y_2) $(x_3, y_3) \ldots\ldots (x_n, y_n)$. At A_1, A_2, $A_3 \ldots\ldots A_n$ are placed particles of weights W_1, W_2, $W_3 \ldots\ldots W_n$.

Let G be the centre of gravity of the particles and let its co-ordinates be $(\overline{x, y})$. *Suppose the plane placed horizontally so that the direction of the weights is* perpendicular to the plane (the plane of the paper).

The C. G. remains unaltered by this process since the centre of a system of parallel forces does not change if the forces undergo equal rotations in the same sense.

The weights W_1, W_2, $W_3 \ldots\ldots W_n$ are ***like parallal forces*** acting vertically downwards and their resultant $(W_1+W_2+W_3+\ldots\ldots W_n)$ in the same direction.

OX being in the horizontal plane, is perpendicular to the direction of the weights and to their resultant.

$\therefore$ *Taking moments about the line* OX and remembering that the sum of the moments of any number of forces about a perpendicular straight line is equal to the moment of their resultant, we have

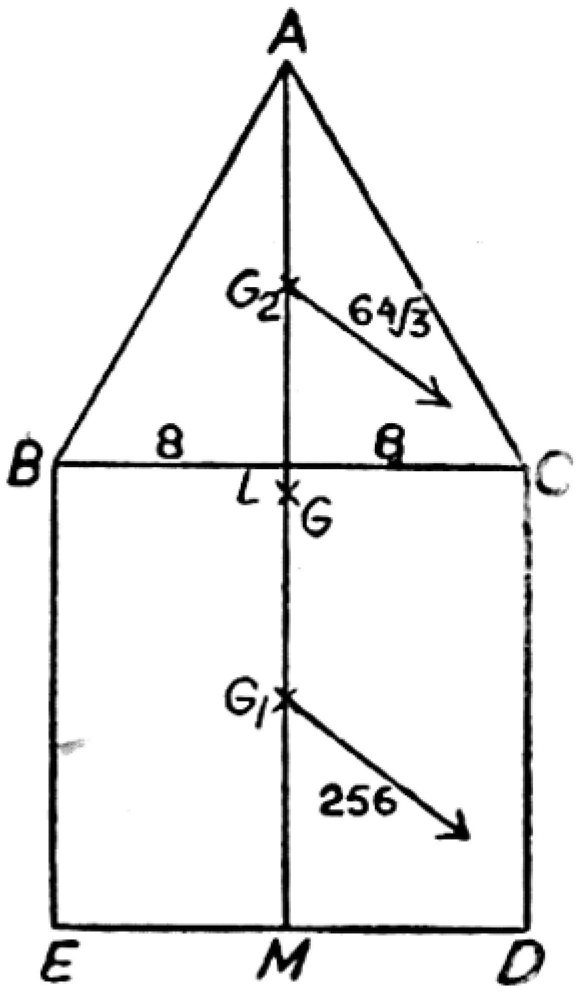
$$\bar{y}\ (W_1+W_2+\ldots\ldots W_n)=W_1y_1+W_2y_2+W_3y_3+\ldots\ldots W_ny_n$$

$$i.e., \ \bar{y} = \frac{W_1y_1+W_2y_2+W_3y_3+\dots\dots+W_ny_n}{W_1+W_2+W_3+\dots\dots+W_n} = \frac{\Sigma W_n y_n}{\Sigma W_n}.$$

Similarly, by taking moments about OY, we get

$$\bar{x} = \frac{W_1y_1+W_2y_2+W_3y_3+W_ny_n}{W_1+W_2+W_3\dots\dots W_n}.$$

Remarks: Observe that in the formulae for C. G. in Arts. 91 and 92, it will make no difference in the position of the centre of gravity if we use quantities *proportional to the weights* instead of the actual weights themselves.



Worked Examples

93. Ex. 1. *An equilateral triangle is described upon one side of a square whose side is 16 inches. Find the distance of the C. G of the figure so formed from the vertex of the triangle.*

The C. G. must be in the line AM, from the symmetry of the figure. LM=16 in.; AL=$8\sqrt{3}$ in.

$\triangle$ ABC=$64\sqrt{3}$ sq. inches; square=256 sq. inch.

The weights are all arranged in one line.

Taking A as origin, we have, by Art. 91.

$$AG = \frac{\Sigma \text{ (each wt. multiplied by its corresponding distance)}}{\text{sum of the weights}}$$

$$= \frac{256\times AG_1+64\sqrt{3}\times AG_2}{256+64\sqrt{3}}$$

$$= \frac{256\times(8+8\sqrt{3})+64\sqrt{3}\times(\frac{2}{3}\times8\sqrt{3})}{256+64\sqrt{3}}$$

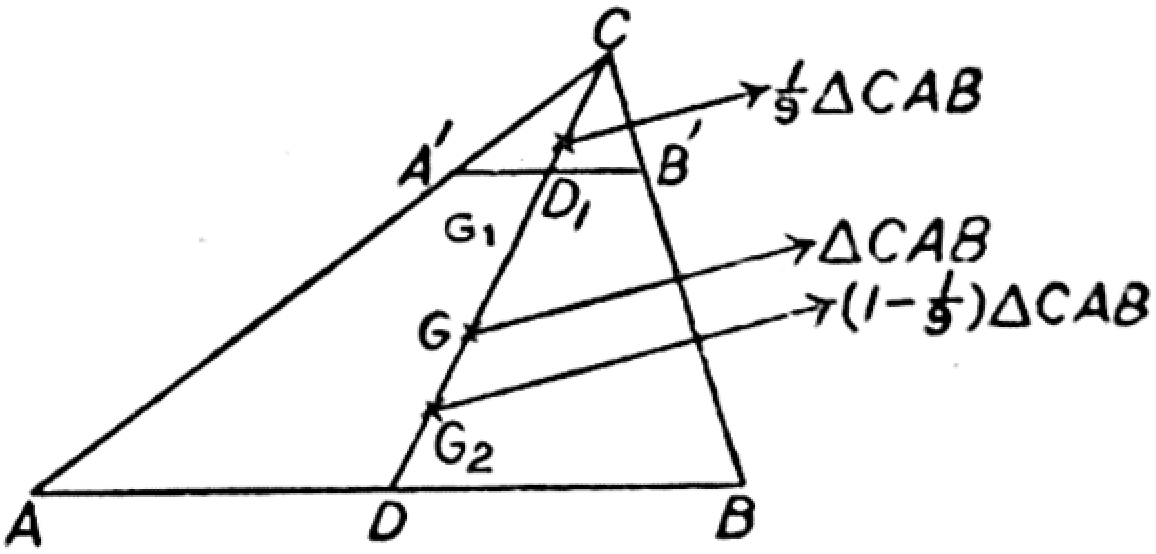
$$= \frac{4(8+8\sqrt{3})+16}{4+\sqrt{3}}$$

$$= \frac{32\times2{\cdot}73+16}{5{\cdot}73} = 18{\cdot}04 \text{ inches.}$$

Ex. 2. *The ninth part of a triangle is cut off towards the vertex by a line parallel to the base. Find the C. G. of the remainder.*

$$\frac{\triangle\ CA'B'}{\triangle\ CAB} = \frac{1}{9} = \frac{CA'^2}{CA^2}, \therefore CA' = \tfrac{1}{3}CA.$$

Also $CD_1 = \frac{1}{3}$ CD, and $DD_1 = \frac{2}{3}$ CD.



$\triangle$ CA′B′ is cut off; we have to find the C. G. of the quadrilateral A′ABB′. If G be the C.G. of $\triangle$CAB, G lies in the median CD and $CG = \frac{2}{3}$ CD. Similarly, if G_1 be the C. G. of $\triangle$CA′B′, G_1 lies in CD_1, and $CG_1 = \frac{2}{3}\ CD_1$. Let G_2 be the C. G. of quad. A′ABB′, the wts. at G and G_1 and G_2 are represented by arrow-heads as shown and they are arranged in the line CD; CG_2 is to be found.

By Art. 91,

$$CG = \tfrac{2}{3}\ CD = \frac{CG_1 \times \text{wt. at } G_1 + CG_2 \times \text{wt. at } G}{\text{wt. at } G_1 + \text{wt. at } G_2}$$

$$= \frac{CG_1 \times \frac{1}{9}\ \triangle\ CAB + CG_2 \times \frac{8}{9}\ \triangle\ CAB}{\triangle\ CAB}$$

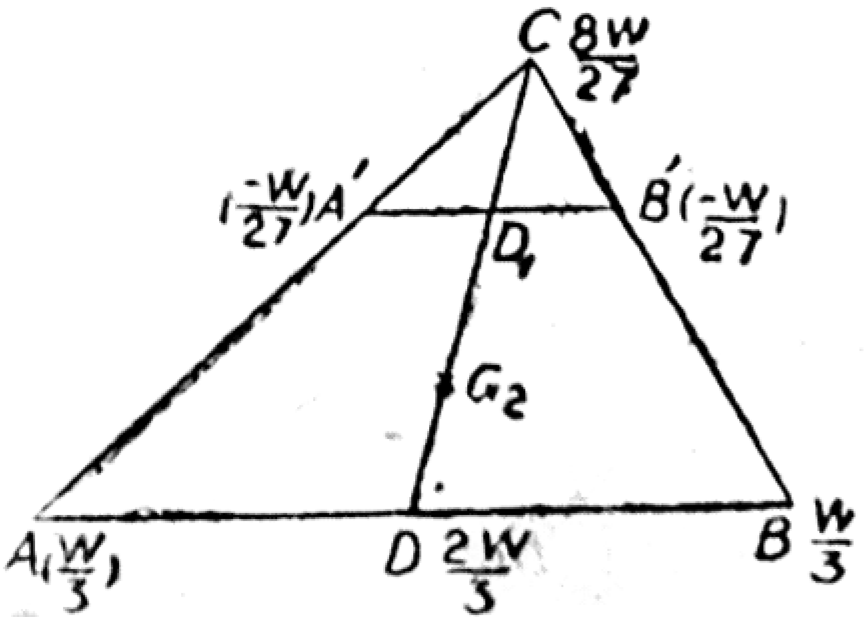
$$= \frac{\frac{2}{3}\ CD_1 \times \frac{1}{9} + CG_2 \times \frac{8}{9}}{1}.$$

$$\therefore \tfrac{8}{9}\ CG_2 = \tfrac{2}{3}CD - \frac{2}{27}CD_1 = \tfrac{2}{3}CD - \frac{2}{27} \times \tfrac{1}{3}CD.$$

$$\therefore\ CG_2 = \tfrac{3}{4}CD - \frac{1}{36}CD = \frac{26}{36}CD = \frac{13}{18}CD$$

$$i.e.\ DG_2 = \frac{5}{18}CD = \frac{5}{18} \text{ of median.}$$

Alternative method. When a portion of a body is removed, the *weight of the portion removed may be taken negatively, and added to the whole.*



The C. G. of the remainder may thus be found in a simpler manner.

The weight W of the whole may be distributed to the three corners, *i.e.* $\frac{W}{8}$ at each of A, B and C. The weight $-\frac{W}{9}$ of *the portion removed* may be taken to be $-\frac{W}{27}$ at each of the corners A′, B′ and C.

Thus we have $\frac{W}{3}-\frac{W}{27}$ or $\frac{8W}{27}$ at C, $-\frac{2W}{27}$ at D_1, and $\frac{2W}{3}$ at D.

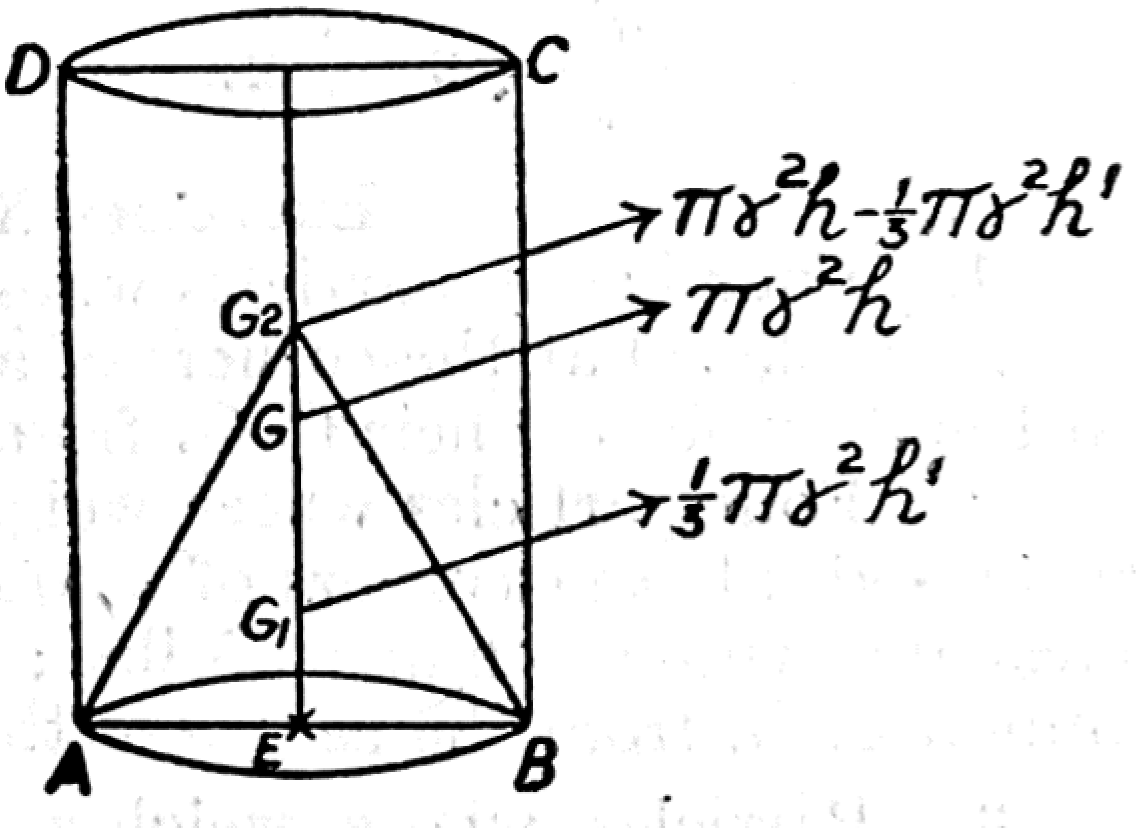
Taking moments about C,

$$\frac{2W}{3}.\ CD-\frac{2W}{27}.\ CD_1+\frac{8W}{27},\ O=\frac{8W}{9}.\ CG_2,$$

$$\text{or } \tfrac{2}{3}\ CD-\frac{2}{27}.\ \tfrac{1}{3}\ CD=\tfrac{8}{9}.\ CG_2,$$

$$\text{Whence } CG_2=\frac{13}{18}CD.$$

Ex 3. *Show how to cut out of a uniform cylinder, a cone whose base coincides with that of the cylinder, so that the centre of gravity of the remaining solid may coincide with the vertex of the cone.*



Let the base of the cylinder whose height is h be a circle of radius r, and let the height of the cone be h'. Let G be the C.G. of the cylinder, and G_1 that of the cone. Then evidently $EG_1=\frac{1}{4}h'$, $EG=\frac{h}{2}$; *EG_2* $(=k')$ *is to be found* where G_2 is, by the question the C.G. of the remaining solid. From symmetry, all the weights are arranged in one line. Therefore, by the theorem of Moments

$$EG=\frac{\Sigma\text{ (each wt.}\times\text{corresponding distance)}}{\text{sum of the weights}}$$

$$=\frac{EG_1\times\frac{1}{3}\pi r^2h'+EG_2(\pi r^2h-\frac{1}{3}\pi r^2h')}{\frac{1}{3}\pi r^2h'+(\pi r^2h-\frac{1}{3}\pi r^2h')}$$

$$= \frac{\frac{1}{4}h' \times \frac{1}{3}\pi r^2 h' + h'(\pi r^2 h - \frac{1}{3}\pi r^2 h')}{\pi r^2 h}$$

i.e. $$\frac{h}{2} = \frac{\frac{1}{4}h' \times \frac{1}{3}h' + h'\left(h - \frac{h'}{3}\right)}{h},$$

or $$\frac{h^3}{2} = \tfrac{1}{12}h'^3 + h'\left(h - \frac{h'}{3}\right) = hh' - \tfrac{1}{4}h'^2.$$

$$\therefore \quad \tfrac{1}{2} = \left(\frac{h'}{h}\right) - \tfrac{1}{4}\left(\frac{h'}{h}\right)$$

i.e. $$\left(\frac{h'}{h}\right)^2 - 4\left(\frac{h'}{h}\right) + 2 = 0.$$

$$\therefore \quad \frac{h'}{h} = \frac{+4 \pm \sqrt{16-8}}{2}$$

$$= 2 \pm \sqrt{2},$$

i.e., $h' = (2+\sqrt{2})h$, or $(2-\sqrt{2})h$.

But h' cannot be greater than h.

$\therefore$ the value $(2+\sqrt{2})$ is rejected.

$$\frac{h'}{h} = 2 - \sqrt{2}.$$

Exercises XIA

1. Four heavy particles whose weights are 3, 5, 4 and 2 lbs., are placed at the corners of a square whose side is a; find the distance of their C. G. from the centre.

2. Four particles whose weights are 4, 6, 5, and 8 lbs. are placed at the corners of a square plate whose side is 26 inches and whose weight is 8 lbs.; find the distance of the common C. G. from the centre of the plate.

3. Particles whose weights are 1, 9, 2, 3, 3 and 6 lbs. respectively, are placed at the angular points taken in order, of a regular hexagon whose side is a; find their C. G. and its distance from the centre of the hexagon.

4. On a uniform straight bar 5 ft. long and of weight 5 lbs., bodies whose weights are 1, 2, 3, 4 lbs. are hung by strings at distances 1, 2, 3, 4 feet respectively from an extremity; find the distance from the centre of the bar of the fulcrum, on which the whole will rest.

5. On a line AB, 12 inches long are placed two particles of masses 1 and 3 lbs. at distances 1 and 3 inches from A and B respectively. Find the position of a third particle of mass 2 lbs. such that the C. G. of the particles may be at the middle point of AB.

6. A uniform rod AB is 4 ft. long and weighs 3 lbs. 1 lb. is then attached to the end A, 2 lbs. at a point distant 1 ft. from A, 3 lbs. at 2 ft. from A, 4 lbs. at 3 ft. from A and 5 lbs. at the end B. Find the distance from A, of the C. G. of the system.

7. A uniform rod AB is 4 feet long and weighs 6 lbs. and weights are attached to it as follows: 1 lb. at A, 2 lbs. at, 1 foot from A, 3 lbs. at 2 ft. from A, 4 lbs. at 3 ft. from A and 5 lbs. at B. Find the distance from A of the C. G. of the system.

8. Weights of 1, 2, 3, 4 lbs. are placed at the corners A, B, C, D respectively of a square ABCD of side 8 inches. Find the distance of the C. G. of the system from AB and AD.

9. Weights of 5, 6, 9 and 7 lbs. are placed at the corners A, B, C, D of a square of side 27 inches. Find the distance of their centre of gravity from A.

10. Weights of 1, 5, 3, 4, 2 and 6 lbs. are placed at the angular points of a regular hexagon taken in order. Show that their centre of gravity is at the centre of the hexagon.

11. Weights of 1, 2, 3, 4, 5 and 6 lbs. are placed at the angular points of a regular hexagon taken in order. Find the distance of their centre of gravity from the centre of the hexagon, the length of the side being 14 inches.

12. Masses weighing 1, 2, 3, 4 lbs. are placed respectively at the corners A, B, C, D of a rectangle; AB = 6 feet, BC = 12 feet. Find the perpendicular distances of the centre of gravity from AB and BC.

13. Four like parallel forces 1 lb., 2 lbs., 3 lbs. and 4 lbs. act at points placed at equal intervals along a line; *show that their centre is unaltered by the removal of the third force.*

14. A uniform triangular lamina is supported at its three angular points; show that whatever be the form of the triangle, the pressures on the supports are equal.

15. Five equal particles are placed at five of the angular points of a regular hexagon. Find the distance of their centre of mass from the centre of the hexagon, the side being a.

16. Seven equal particles are placed at seven of the angular points of a cube whose diagonal is a. Find the distance of their centre of inertia from the unoccupied angle.

17. Having given the positions of three particles P, Q, R, and the positions of centres of gravity of P and Q and of P and R, find the centre of gravity of Q and R.

[Let P, Q, R be placed at the vertices A, B, C; let G be the C.G. of the 3 particles. Then C.G. of Q, R is the point where the line AG cuts the side BC.]

18. *If a quadrilateral be such that one of its diagonals divides it into two equal triangles, show that the centre of gravity of the quadrilateral lies in that diagonal.* (*14 Ex. XI*)

19. How would you place masses 1, 1 and $\sqrt{2}$ at circumference of a circle, so that their centre of gravity should fall at the centre.

[If the particles be joined to form a triangle, the sides subtend the angles 135°, 135°, 90° at the centre].

20. If the centre of gravity of three particles placed one at each of the angular points of a triangle coincide with the centre of gravity of the triangle, prove that the particles are equal.

21. If a given weight be placed anywhere on a given heavy uniform triangular lamina, show that the centre of gravity of the system will be within a certain triangle.

22. Show how to place three given masses on a circle so that its centre may be their centre of mass. What conditions must the masses satisfy in order that it may be possible? (*P. U. 1908*)

23. A triangular lamina whose sides are in the ratio 3 : 4 : 5, is suspended from the middle point of the longest side. Find its inclination to the vertical. (*P. U. 1907*)

24. A triangular lamina whose sides are 7, 24, 25 inches is suspended from the vertex opposite to the longest side. Find the inclination of this side to the vertical in the position of equilibrium.

25. ABCD is a square, O the centre. If △ AOB *is removed* find the C. G. of the remainder.

26. Find the C. G. of a plane area formed by an equilateral triangle and a square of side *a*, the base of the triangle coinciding with one of the sides of the square.

27. The sides of a triangular area are 3, 4 and 5 inches, and its *in-circle is removed*; find the distance of the C. G. of the remainder from the sides 3 and 4.

28. If three parallel forces P, Q, R act at A, B, C the angular points of a triangle ABC, find the condition that their centre falls at (*i*) the in-centre (*ii*) an ecentre (*iii*) the circum-centre (*iv*) the centroid (*v*) the orthocentre.

29. On the side AB of an equilateral triangle ABC and on the side remote from C is described a rectangle whose height is ½ AB ; prove that the C. G. of the whole figure thus formed is the middle point of AB. (*P. U. 1909*)

30. ABCD is a square of paper, and E, F are the middle points of AB, AD. Find the centre of gravity of the portion left when △ *AEF is cut off* and the centre of gravity of the whole when △ AEF is doubled over.

31. A uniform wire is bent into the form of three sides of an equilateral polygon. If the sides be AB, BC, CD, and the centre of gravity of the wire be the intersection of AC and BD, show that the polygon is a regular hexagon.

32. Show by integration or otherwise that the centre of gravity of a circular lamina or ring is at its centre.

33. In the trapezium ABCD, the parallel sides AB and CD are of lengths *a* and *b*, respectively. Find the position of its C. G. Also if this C. G. is the same as that of particles *m*, *m*, *m'*, *m'* placed at A, B, C, D, respectively, find the ratio $m : m'$. (*P. U. 1915*)

34. *A quadrilateral lamina has its C. G. at the intersection of its diagonals, show that it must be parallelogram.* [*15 Ex. XI*] (*P. U. 1912*)

35. If (x_1, y_1), (x_2, y_2), (x_3, y_3) be the co-ordinates of the vertices A, B, C of a triangle ABC, show that the co-ordinates of its C. G. are $\frac{1}{3}(x_1+x_2+x_3)$, $\frac{1}{3}(y_1+y_2+y_3)$.

Hint : The C. G. of the triangle is the same as that of three equal masses *m*, *m*, *m* placed at A, B, C. Apply Art. 92.

36. From a body of weight W, a piece, of weight w, is cut off and moved to a distance x, show that the centre of gravity of the whole is thereby moved to a distance $\frac{x\,w}{W}$ [*P. U. 1935*].

37. From a uniform circular disc of radius r, and centre O is cut out a circle whose centre is G_1 and which passes through O and whose diameter is equal to the radius of the disc. Find G_2, the C.G. of the remainder.

38. From a circular plate, a square whose vertices lie on the circumference, is removed. Show that the centre of gravity of the remainder of the plate is the same as that of the whole plate. [See Note 1. Art. 81].

39. Show that if a circular hole be drilled in a uniform triangular lamina with its centre at the C.G of the $\triangle$, the C. G. of the lamina will remain unaltered.

Hint : Proceed as in Ex. 38.

40. Where must a circular whole of 1 foot radius, be punched out of a circular disc, of 3 feet radius so that the centre of gravity of the remainder may be 2 inches from the centre of the disc. ?

41. On the same base BC are two triangles ABC, A′BC, the vertex A′ falling within the former triangle. Find the position of A′ when it is the centre of gravity of the area between the two triangles.

42. A square plate of 10 inches side has a circular portion removed from it, the boundary of the removed portion touching a side of the square and passing through its centre. If the line joining their side is perpendicular to a side of the square, find the C.G. of the plate.

43. ABCD is a trapezium whose parallel sides AB, CD are of lengths a and b and h is the perpendicular distance between them. If X, Y denote the distances of the C.G. of the figure from a and b respectively, prove that

$$(i)\ X=\frac{h}{3}\ \frac{2b+a}{a+b} \qquad (ii)\ Y=\frac{h}{3}\ \frac{2a+b}{a+b}.$$

CHAPTER VIII

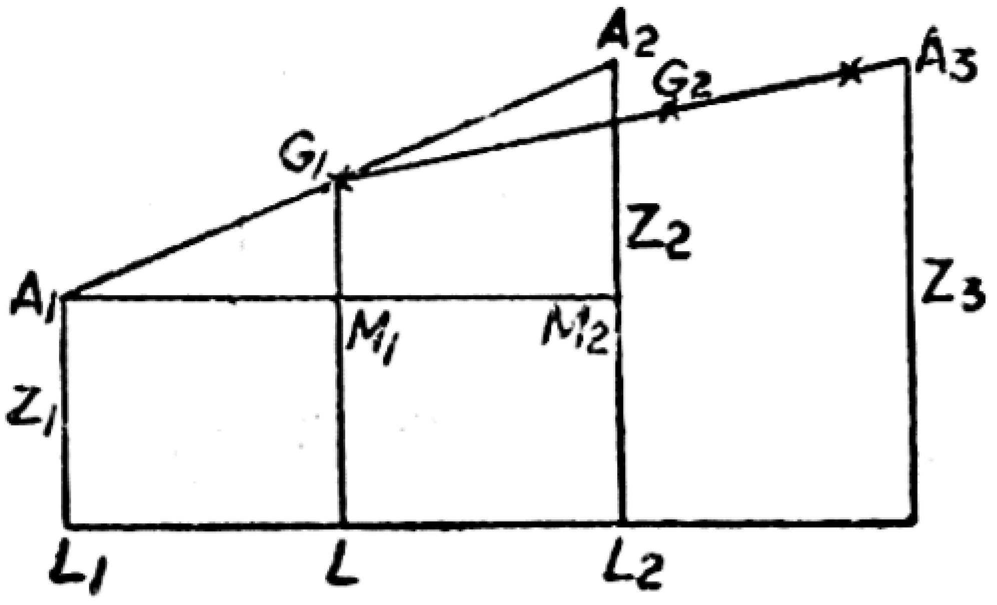
DETERMINATION OF C.G. BY AID OF INTEGRAL CALCULUS

94. If W_1, W_2, W_3......be the weights of a number of particles placed at points A_1, A_2, A_3......in space whose distances from a fixed plane are z_1, z_2, z_3......respectively, then

$$\bar{z}=\frac{W_1z_1+W_2z_2+W_3z_3+\ldots\ldots}{W_1+W_2+W_3\ldots\ldots}=\frac{\Sigma Wz}{\Sigma W},$$

where $\bar{z}$ is the distance of the C. G of the particles from the same plane.

Let G_1 be the C. G. of W_1 at A_1 and W_2 at A_2, and $G_1L=z'_1$; draw line $A_1M_1M_2$ parallel to the plane. Then



$$\frac{W_2}{W_1}=\frac{A_1G_1}{G_1A_2}=\frac{G_1M_1}{A_2M_2}$$

$$=\frac{z'_1-z_1}{z'_2-z_1}$$

$$\therefore\quad z'_1=\frac{W_1z_1+W_2z_2}{W_1+W_2}.$$

Again, if z'_2 be the distance from the plane of the C. G. of W_1+W_2 at G_1 and W_3 at A_3, we have, similarly,

$$z'_2=\frac{(W_1+W_2)z'_1+W_3z_3}{(W_1+W_2)+W_3}$$

$$=\frac{W_1z_1+W_2z_2+W_3z_3}{W_1+W_2+W_3}.$$

Proceeding in this way till all the weights are exhausted, we get, if $\bar{z}$ be distance of the final C.G. from the plane,

$$\bar{z}=\frac{W_1z_1+W_2z_2+W_3z_3+\ldots}{W_1+W_2+W_3+\ldots}=\frac{\Sigma W_1z_1}{\Sigma W_1}$$

Corollary. If YOZ, ZOX, XOY be three planes at right angles to each other, the co-ordinates $\bar{x}$, $\bar{y}$, $\bar{z}$ and the final resultant are easily shown in the following orm :—

$$\bar{x}=\frac{W_1x_1+W_2x_2+W_3x_3+\dots}{W_1+W_2+W_3+\dots}=\frac{\Sigma W_r x_r}{\Sigma W_r},$$

$$\bar{y}=\frac{W_1y_1+W_2y_2+W_3y_3+\dots}{W_1+W_2+W_3+\dots}=\frac{\Sigma W_r y_r}{\Sigma W_r},$$

$$\bar{z}=\frac{W_1z_1+W_2z_2+W_3z_3+\dots}{W_1+W_2+W_3+\dots}=\frac{\Sigma W_r z_r}{\Sigma W_r}.$$

Final resultant $=\Sigma W_r$.

95. If δv is an elementary volume of a body whose *uniform density* is ρ, the mass of the corresponding small element at the point (x, y, z) is $\rho\,\delta v$. Hence, from the previous article, we have, for the co-ordinates of the C. G.

$$\bar{x}=\frac{\Sigma g\rho\,\delta v\,x}{\Sigma g\rho\,\delta v}.$$

$$=\frac{\int x.\,g\rho dv}{\int g\rho dv}$$ in the notation of Integral Calculus

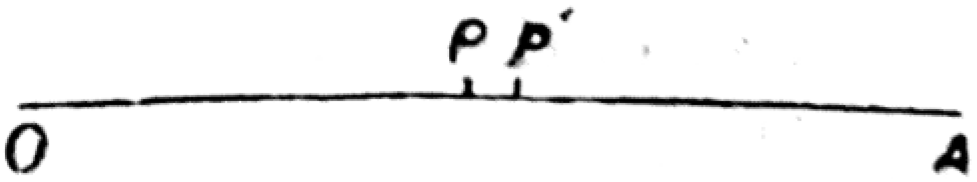
i.e.
$$\bar{x}=\frac{\int x dv}{\int dv}.$$

Similarly,
$$\bar{y}=\frac{\int y dv}{\int dv},\qquad \bar{z}=\frac{\int z dv}{\int dv}.$$

These are also the *co-ordinates of the centre of inertia.*

Determination of the centre of gravity.

96. (*a*) *To find the centre of gravity of a uniform* **straight wire.**



Let OA be the wire of sectional area k and length $2a$. Take OA as x-axis and O as origin.

The element PP′ of the wire $=k\delta x$, where OP $=x$.

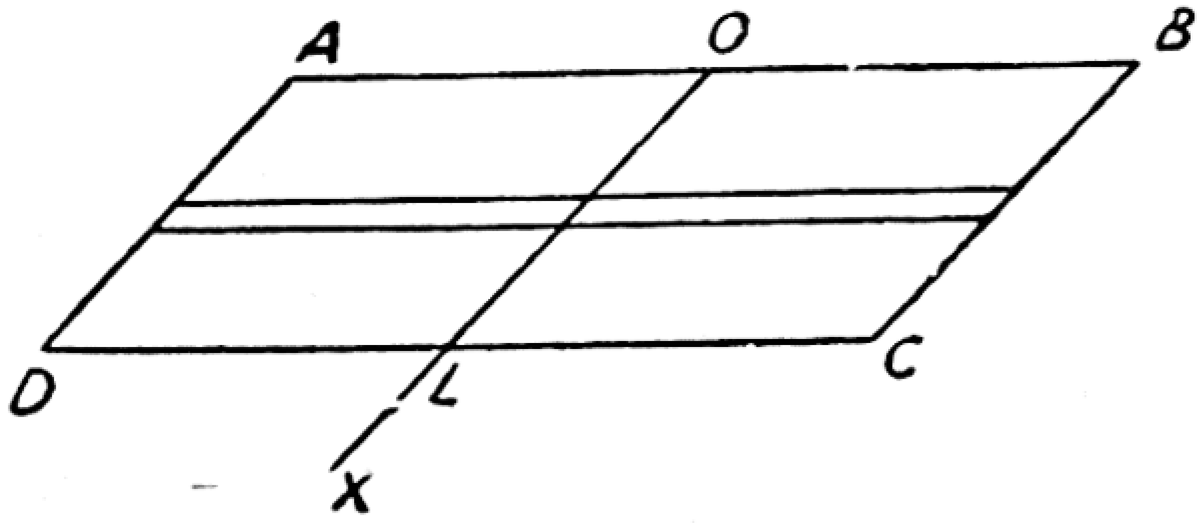
If G be the C.G. at distance $\bar{x}$ from O, then

$$\bar{x} = \frac{\Sigma k\delta x.x}{\Sigma k\delta x} = \frac{\int_0^{2a} x\,dx}{\int_0^{2a} dx} = \frac{2a^2}{2a} = a.$$

Hence C. G. is at the middle point.

(*b*) *To find the centre of gravity of a* **parallelogram** *of uniform thickness and density.*

Let O and L be the middle points of AB and CD of the parallelogram ABCD. Then OL is evidently a line of symmetry. Take O as origin and OL as x-axis. Divide the parallelogram into strips parallel to AB. The area of a strip at distance x from O measured along OX varies as the width, or as δx.



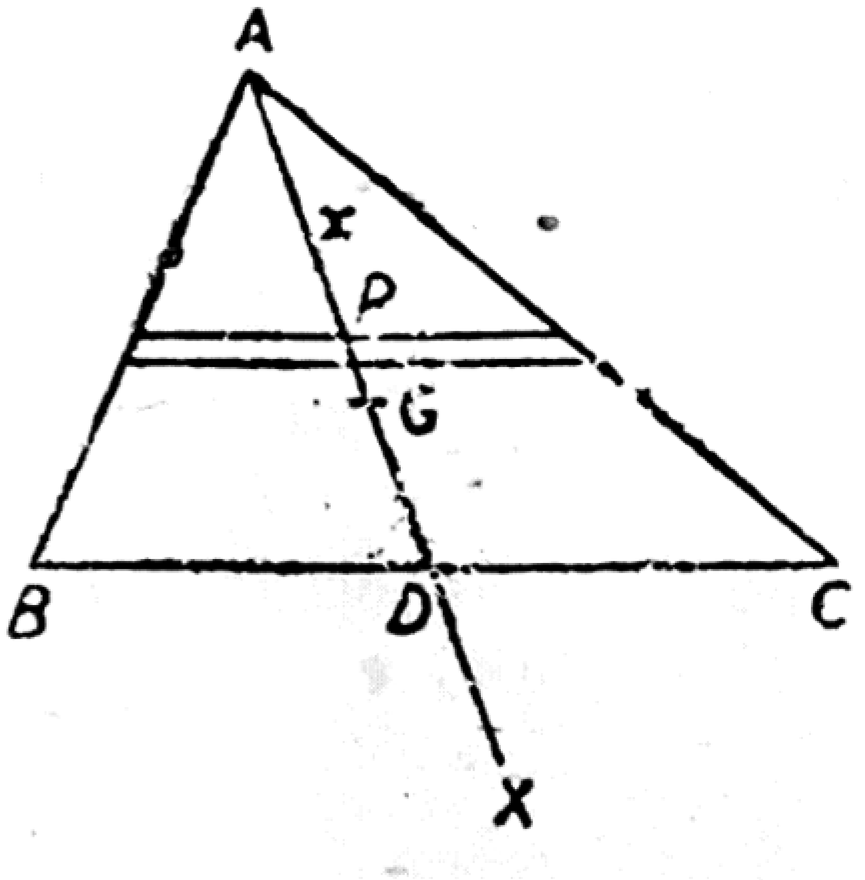
$$\therefore\ \bar{x} = \frac{\int_0^{2b} x\,dx}{\int_0^{2b} dx} = b, \text{ where BC} = 2b.$$

Thus the C. G. of the parallelogram is at the intersection of the line joining the mid-points of the opposite sides.

The result can also be deduced from (*a*).

(*c*) *To find the centre of gravity of a* **triangular lamina** *of uniform thickness k and density ρ.*

Let AD be the median of △ ABC. Then AD is a line of symmerty. Take A as origin and AD as x-axis.



Divide the triangle into strips parallel to the base BC. Consider any strip at a distance x measured along AD. The width of the strip is proportional to δx, the length intercepted on AD and the length of the strip is proportional to x.

Thus the mass of the strip is proportional to $k\rho x\delta x$.

$$\therefore \quad \overline{\mathbf{x}} = \frac{\Sigma k\rho x\delta x . x}{\Sigma k\rho x\delta x} = \frac{\int_0^{AD} x^2 dx}{\int_0^{AD} x dx} = \tfrac{2}{3}\mathbf{AD}.$$

If G be the C. G. of the lamina, AG $= \frac{2}{3}$ AD.

The result can be deduced from (*a*).

Note.—It is obvious that the centre of gravity of a triangle coincides with the centroid of the triangle (see Art. 35). On this account, this point is called centroid of the triangle also. It is the C. G. of three equal particles placed one at each angular point, for if M be the mass of each particle, the C.G. of the particles at B and C is at D. Suppose them placed at D. The C. G. of two particles—one of mass 2M at D and the other of mass M at A—divides AD at G′ such that AG′ : G′D = 2 : 1. Thus G′ coincides with G.

(*d*) *To find the centre of gravity of a uniform tetrahedron or a pyramid on a triangular base.*

We have shown that the C.G. of the tetrahedron lies on OL. Take O as origin and OL as x-axis. Consider the lamina lying between two parallel planes A′B′C′ and A″B″C″ meeting OL in L′ and L″. Let OL′ $= x$, OL″ $= x + \delta x$. Then the thickness of the lamina is proportional to δx.

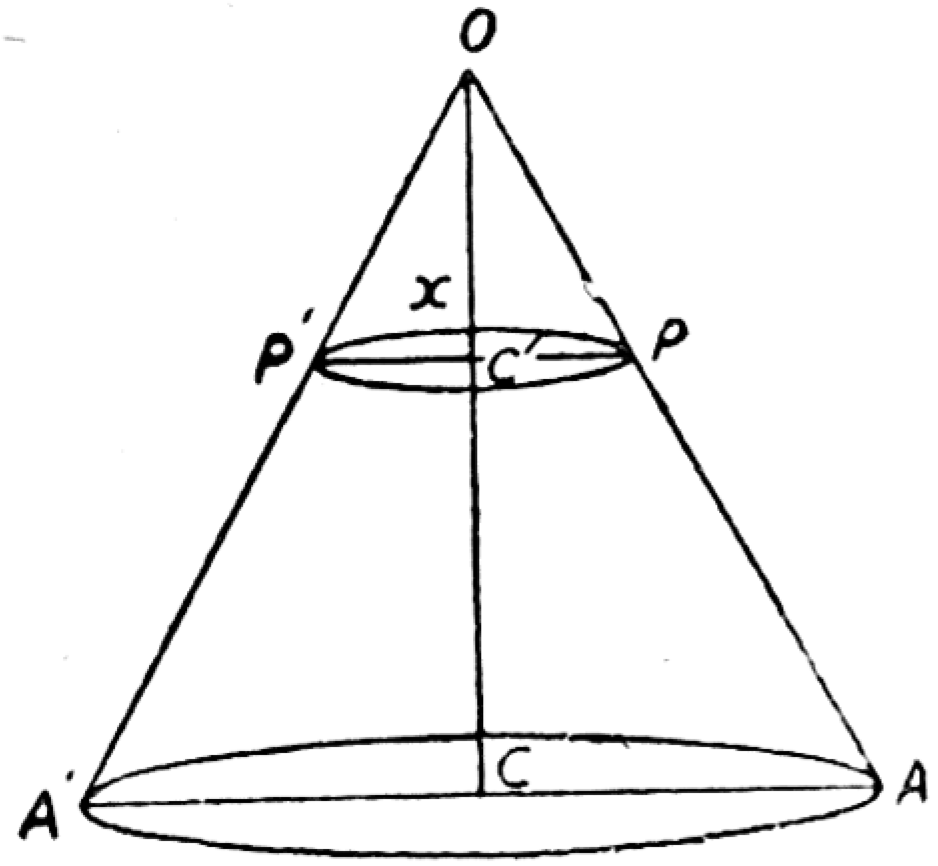
$\because$ △ A′B′C′ : △ ABC = OL′^{2} : OL′,

$\therefore$ the area of the lamina is proportional to x^2

$$\therefore \quad \mathbf{x} = \frac{\int_0^{OL} x^2\, dx \,.\, x}{\int_0^{OL} x^2 dx} = \tfrac{3}{4}\ \mathbf{OL}.$$

(*e*) *Analytical determination of the centre of gravity of a* **solid cone.**

Take O the vertex and OC the axis, as origin and x-axis respectively, C being the centre of the base.



If a be the radius of the base and h the height of the cone, the radius of any circular section or slice parallel to the base at a distance x from O is equal to $\frac{xa}{h}$.

If δx be the thickness and ρ the uniform density of that cone, the mass of the slice $= \rho\pi \frac{a^2x^2}{h^2}.\delta x$ to the 1st. approximation.

$$\therefore \quad \bar{x} = \frac{\int_0^h \rho\pi \frac{a^2x^2}{h^2} dx.x}{\int_0^h \rho\pi \frac{a^2x^2}{h^2}\, dx} = \frac{\int x^3\, dx}{\int x^2\, dx} = \tfrac{3}{4}\mathbf{h}.$$

(*f*) *Analytical determination of the* **C. G.** *of the surface of a* **hollow cone.**

Take the vertex O as origin and the axis of the line as x-axis. Let h be its height and a the radius of the base.

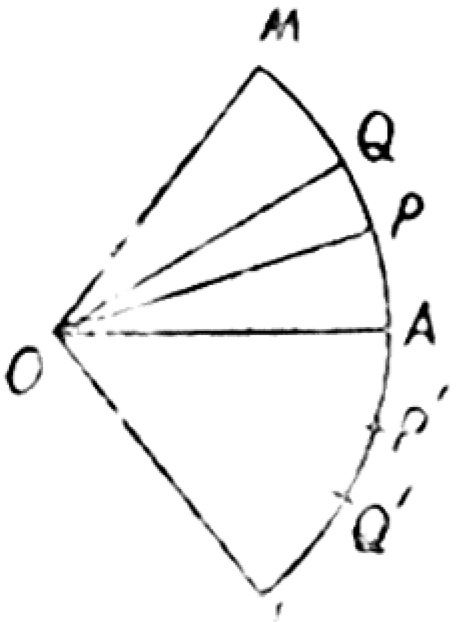
The circumference of a circular ring at a distance x is equal to $2\pi \frac{a}{h}x$.

The area of an elementary surface between two rings at distance δx is proportional to $2\pi \frac{a}{h} x\delta x$.

$$\therefore \quad \bar{\mathbf{x}} = \frac{\int_0^h 2\pi \frac{a}{h} x\, dx.x}{\int_0^h 2\pi \frac{a}{h} x.\, dx} = \frac{\int x^2\, dx}{\int x\, dx} = \tfrac{2}{3}\mathbf{h}.$$

97. *To find the centre of gravity of a* **circular arc.**

Let the arc LM subtend an angle 2α at the centre O and let OA bisect the arc.



Take the origin at O and OA as x-axis. OA is an axis of symmetry. Consider a small element PQ of half the arc AM. Let $\angle AOP=\theta$ and $\angle AOQ=\theta+\delta\theta$.

If k denote the area of a transverse section, a the radius, and ρ the density, the mass of this element $=\rho ka\ \delta\theta$.

There is a similar element P′Q′ in the other half AL.

Thus we have two equal particles equidistant from the central line. Their centre of gravity lies on OA at distance $a\ \mathrm{cose}\ \theta$ from O.

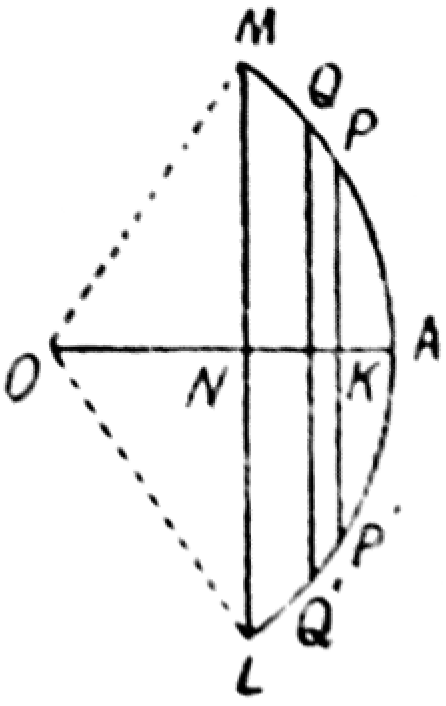
Hence the centre of gravity of the arc is given by

$$\bar{x}=\frac{\Sigma mx}{\Sigma m}=\frac{\int_0^\alpha 2\,\rho ka d\,\theta.\ a\cos\theta}{\int_0^\alpha 2\,\rho ak d\,\theta}=\frac{a\int\cos\theta\,d\theta}{\int d\,\theta}=a\,\frac{\sin\alpha}{\alpha}$$

For a *semicircular arc* $\bar{x}=\frac{2a}{\pi}\left(\alpha=\frac{\pi}{2}\right)$

98. To find the centre of gravity of a **segment of a circle.**

Let LAMNL be the segment, O the centre and OA bisect the arc LM.



Let $\angle LOM=2\alpha$.

Take the origin at O and OA as x-axis.

OA is an axis of symmetry.

Divide the segment by strips parallel to the chord LNM.

Let PQQ′P′ be one of them.

Let $\angle AOP=\theta$, $\angle AOQ=\theta+\delta\theta$.

Then $PP'=2a\sin\theta$

and the width of the strip $=PQ\sin\theta=a\delta\theta.\ \sin\theta$.

If ρ be the density and k the thickness, the mass of the strip is $2k\rho a^2 \sin^2\theta\delta\theta$ and it has its C. G. on OA at K, where $OK = a\cos\theta$.

Thus the centre of gravity of the segment is given by

$$\bar{x} = \frac{\int_0^\alpha 2\rho a^2 \sin^2\theta,\ d\theta\ a\cos\theta}{\int_0^\alpha 2k\,\rho a^2 \sin^2\theta\, d\theta} = \frac{\int_0^\alpha a\sin^2\theta\cos\theta\, d\theta}{\int_0^\alpha \sin^2\theta\, d\theta}$$

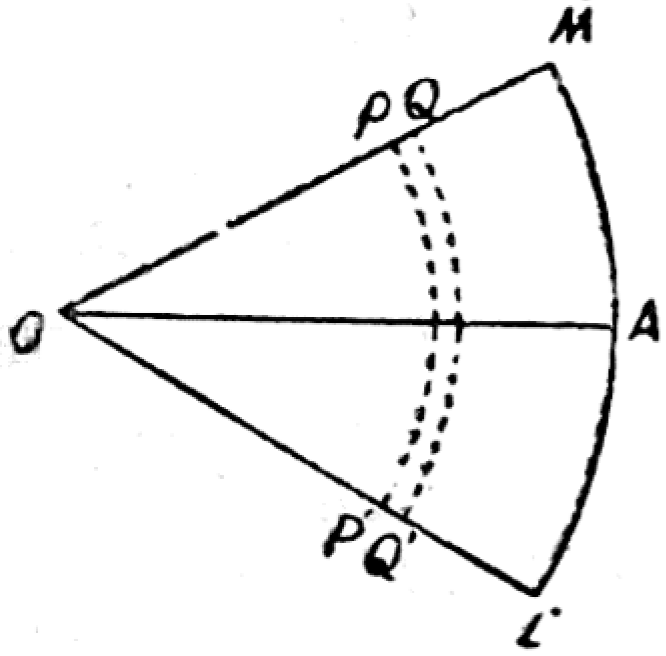
$$= a\,\frac{\frac{1}{3}\sin^3\alpha}{\frac{1}{2}(\alpha-\sin\alpha\cos\alpha)} = \frac{2}{3}a\,\frac{\sin^3\alpha}{\alpha-\sin\alpha\cos\alpha}.$$

It follows that the *centre of gravity of a semi-circle is at a distance* $\frac{4a}{3\pi}$ *from the centre.*

99. *To find the centre of gravity of a* **sector of a circle.**

Let OLM be the sector of radius a.

$\angle LOM = 2\alpha$, A the middle point of the arc LM.



OA is an axis of symmetry.

Draw two concentric arcs PP' and QQ' of radii r and $r+\delta r$.

The area of the circular strip PQQ'P

$=\alpha[r+\delta r)^2-r^2]$,

and its C. G. lies on OA at distance

$$\frac{r\sin\alpha}{\alpha}.$$

If k be the thickness and ρ the density, then the C. G. of the sector is given by

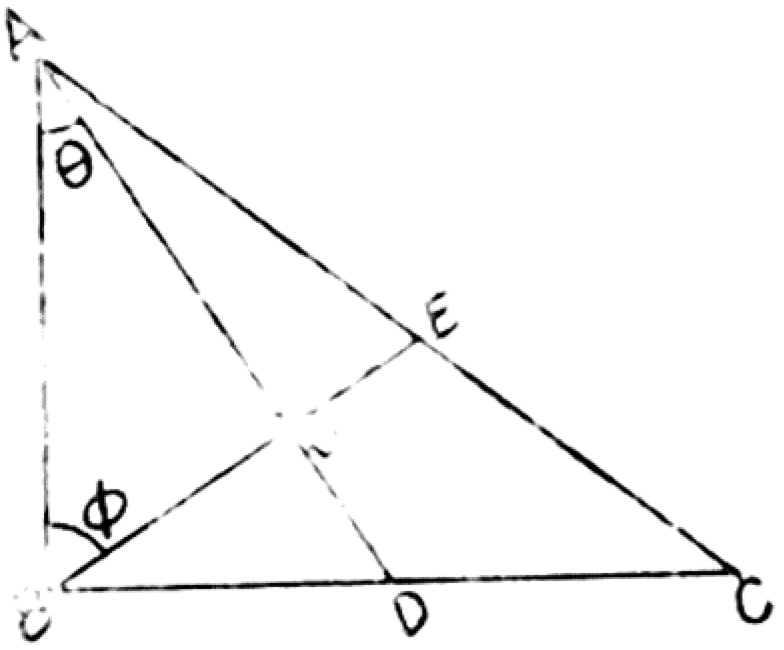
$$\bar{x} = \frac{\int_0^a k\rho 2\alpha r dr\ \frac{\sin\alpha}{\alpha}}{\int_0^a k\rho 2\alpha r dr} = \frac{\sin\alpha}{\alpha}\,\frac{\int_0^a r^2 dr}{\int_0^a r dr} = \frac{2a}{3}\,\frac{\sin\alpha}{\alpha}.$$

Corollaries.

1. For a *semi-circle*, $\bar{x} = \frac{4}{3\pi}a.\ \left(\alpha = \frac{\pi}{2}\right)$

2. For *a* *quadrant* $\overline{x} = \frac{4\sqrt{2}}{3\pi} a \left(\alpha = \frac{\pi}{4} \right)$.

100. Ex. 1. *A heavy triangle ABC is suspended successively from the angles A and B, and two positions of any side are found to be at right angles to each other, prove that* $5c^2 = a^2 + b^2$. (*P. U. 1910*)



Let AD and BE be the medians.

In the first case, for equilibrium AD is vertical.

Let AB make an angle θ with the vertical.

In the second case, for equilibrium BE is vertical.

Let BA make an angle ϕ with the vertical.

Then $\theta + \phi = \frac{\pi}{2}$ (*By the question*).

$\therefore$ $\triangle$AGB is a right angled triangle.

Hence $AG^2 + BG^3 = AB^2$,

$\therefore \frac{4}{9}AD^2 + \frac{4}{9}BE^2 = AB^2$

i.e. $\frac{2}{9}(AB^2 + AC^2 - 2BD^3) + \frac{2}{9}(AB^2 + BC^2 - 2AE^2) = AB^3$,

i.e. $2(c^3 + b^2 - \frac{1}{2}a^2) + 2(c^2 + a^2 - \frac{1}{2}b^2) = 9c^2$,

i.e. $5c^2 = a^2 + b^2$.

Ex. 2. *Prove that if equal triangles be cut from the corners of a given triangle by lines parallel to the respective opposite sides, the centre of mass of the remainder will coincide with that of the triangle.*

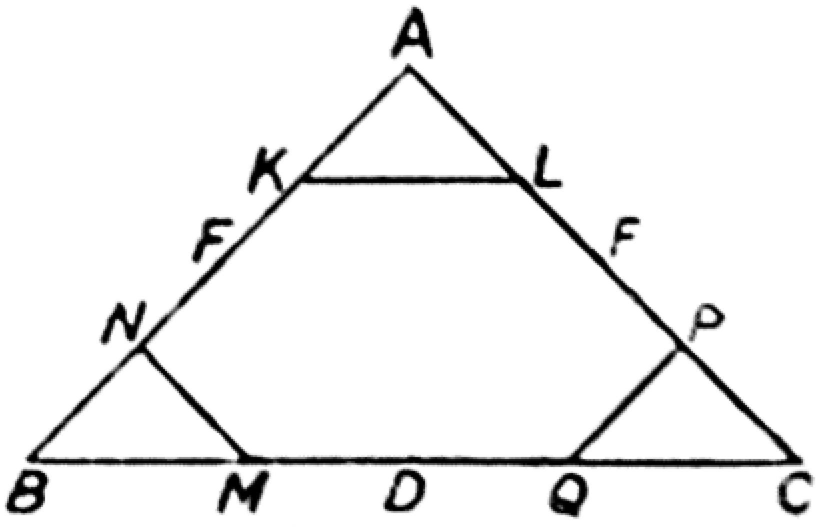
$\triangle$s AKL, BMN, CPQ are equal, each of mass m (suppose).

It is given that KL$\|$BC, MN$\|$CA, PQ$\|$AB

$\therefore$ these triangles are similar, and hence equal in all respects.

$\therefore$ AK=NB, BM=QC, CP=LA.

Thus the middle points of BC, MQ; of CA, PL; of AB, KN are the same.



Replace each of these triangles by equal masses, each $= \frac{m}{3}$, placed one at each vertex.

We thus have $\frac{m}{3}$ at A, $\frac{m}{3}$ at K, $\frac{m}{3}$ at L, $\frac{m}{3}$ at B etc. The centre of mass of $\frac{m}{3}$ at K and $\frac{m}{3}$ at N is at F, the middle point of AB.

The centre of mass of $\frac{m}{3}$ at M and $\frac{m}{3}$ at Q is at D, the middle point of BC.

The centre of mass of $\frac{m}{3}$ at P and $\frac{m}{3}$ at L is at E, the middle point of CA.

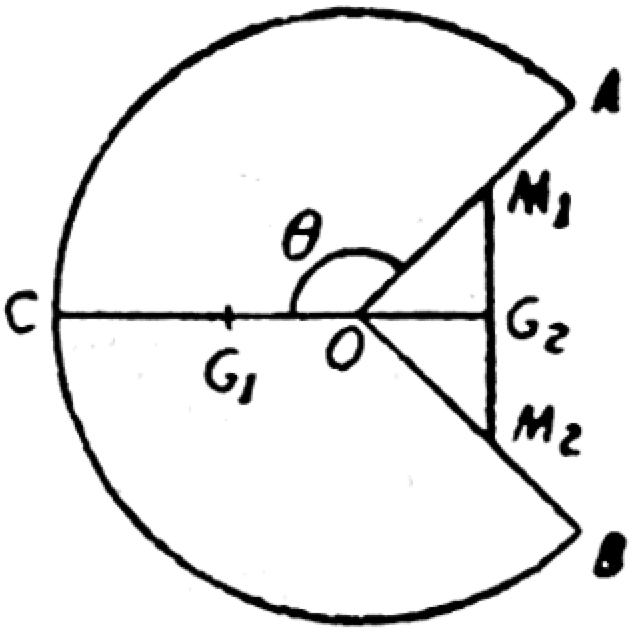
Now the centre of mass of equal particles at D, E, F is the same as the centre of mass of equal particles at A, B, C.

Thus the *centre of mass of the removed portion is the centre of mass of* $\triangle$ *ABC*.

Hence *the centre of mass of the remainder is also the centre of mass of* $\triangle$ *ABC*.

Ex. 3. *A uniform wire is bent into the form of a circular arc and the two bounding radii, and the centre of gravity of the whole is found to be at the centre. Show that the angle subtended by the arc at the centre is* $\tan^{-1}(-\frac{4}{3})$.

Let 2θ be the angle subtended by the arc ABC at the centre O, OA, OB being the bounding radii.



Let OA$=a$ and arc AC$=$arc CB. The C.G. of the arc ACB is at G_1, where $OG_1 = a\frac{\sin\theta}{\theta}$.

The C. G. of the radii OA, and OB is at G_2, where the line joining the middle points M_1, M_2 of OA, OB meets OC produced. Two parallel forces—one proportional to $2a\theta$ and acting at G_1, and the other proportional to $2a$ and acting at G_2—have their centre at O.

∴ Taking moments about O

$$2a\theta.\frac{a\sin\theta}{\theta}=2a.\frac{a}{2}\cos(\pi-\theta). \quad \therefore \tan\theta=-\tfrac{1}{2}.$$

$$\therefore \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = -\tfrac{4}{3} \text{ i.e., } 2\theta = \tan^{-1}(-\tfrac{4}{3}).$$

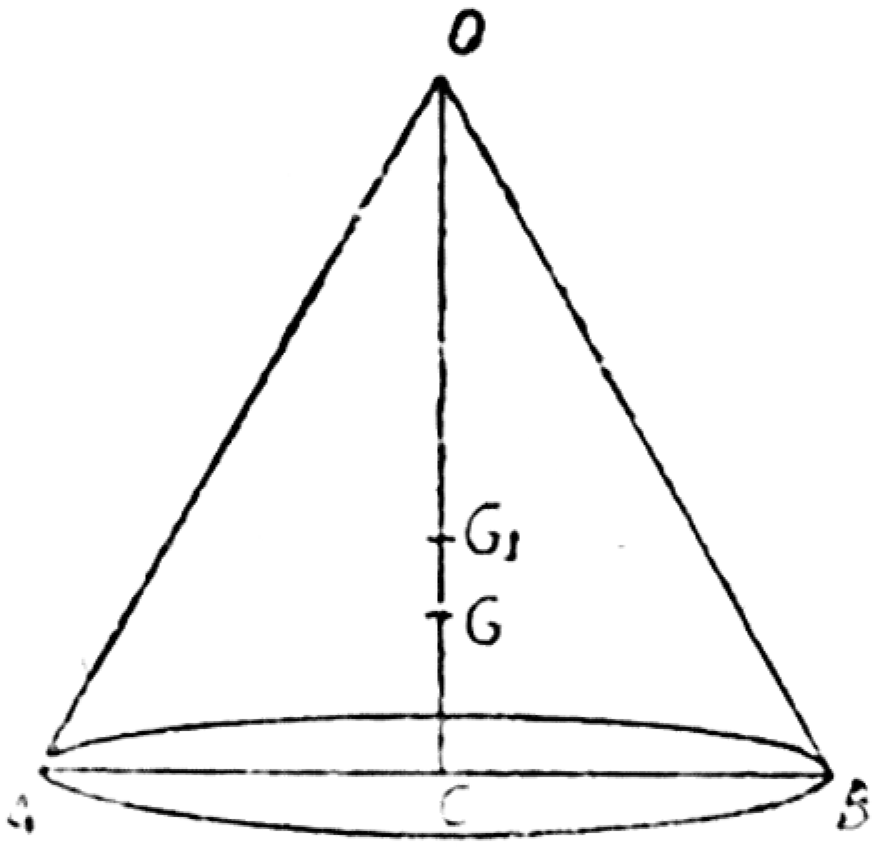
Ex. 4. *A closed vessel made of thin metal sheet of uniform thickness is in the form of a right circular cone. The base is of radius a and the height of the cone is h. Show that the centre of gravity of the vessel is on its axis at distance*

$$\frac{1}{3}\,\frac{h\sqrt{h^2+a^2}}{a+\sqrt{h^2+a^2}}$$

from the centre of the base.

Let C be the centre of the base and O the vertex, G_1 the C. G. of the curved surface,

then $CG_1 = \frac{1}{3}h$.



If ρ be the density of the sheet and k the thickness, the mass of the curved surface $= \rho k \pi a \sqrt{h^2+a^3}$, acting at G_1.

The mass of the base $= \rho k \pi a^3$ acting at C.

Let G be C. G. of the vessel; taking moments about C, we get

$$CG \times \left\{ \pi\rho k a^2 + \pi\rho k a\sqrt{h^2+a^2} \right\}$$

$$= CG_1 \times \pi\rho k a\sqrt{h^2+a^2} + 0.\pi\rho k a^3$$

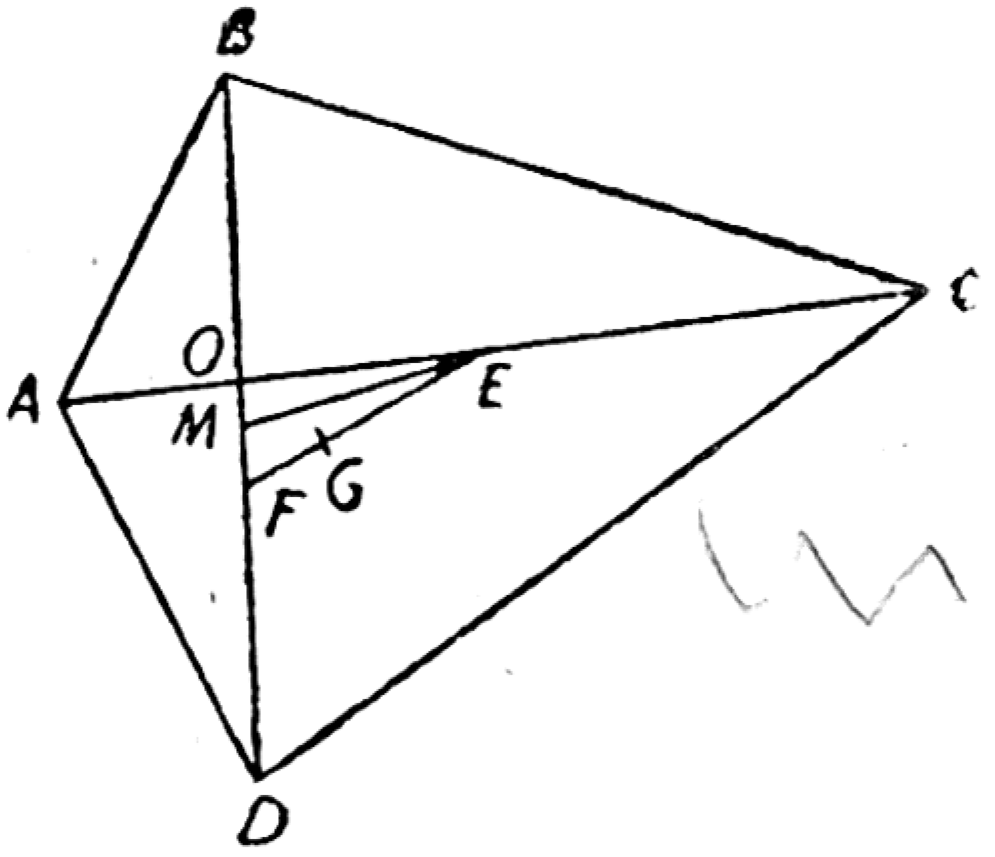
$$\therefore CG = \tfrac{1}{3}h\,\frac{\sqrt{h^2+a^2}}{a+\sqrt{h^3+a^2}}.$$

Ex. 5. *Prove that the centre of gravity of a quadrilateral cannot coincide with the centre of gravity of four equal particles placed at its angular points, unless the quadrilateral is a parallelogram.*

(*i*) Let $3m$ and $3m'$ be the masses of $\triangle$s ABC and ACD.

The C. G. of $\triangle$ABC is the C. G. of the equal particles each $= m$ placed one at each of A, B, C.

The C. G. of $\triangle$ACD is the C. G. of equal particles each $= m'$ placed one at each of A, C, D.



We thus have $m+m'$ at each of A and C, m at B, and m' at D.

The C. G. of $m+m'$ at A and $m+m'$ at C is *at the middle point E of AC.*

The C. G. of m at B and m at D is at F on BD, such that $m.\text{BF}=m'.\text{DF}$..(1)

Thus the C. G. of the whole system is in EF at G, such that FG$=$2GE.

(*ii*) If equal particles of any mass be placed, one at each of A, B, C, D, and M be the middle point of BD, the C. G. of the particles is in EM.

In order that the centres of gravity of both the systems may coincide, it is necessary that EF and EM should coincide.

But this will be possible only, if $m=m'$ [from (1)], *i.e.* if AC bisects the quadrilateral.

Now when AC bisects the quadrilateral, it bisects BD also.

Similarly, BD bisects AC.

Thus the *centres of gravity of both the systems will coincide only if the diagonals of the quadrilateral bisect each other, i.e. if the quadrilateral is a parallelogram.*

Exercises XII

1. ABC is a lamina in the form of an equilateral triangle, each side being 4″ in length; from the corners B, C, two equilateral triangles are cut off. each having its side″ in length. Prove that the distance from A of the C. G. of the remainder is $\frac{53}{14\sqrt{3}}$ inches. (*P. U. 1913*)

2. A cylinder of metal is 2 ft. high, $1\frac{1}{2}$ ft. external diameter and $1\frac{1}{3}$ ft. internal diameter, and $1\frac{2}{3}$ ft. deep inside. It is open at the top. Find the position of its centre of gravity.

3. A square uniform plate is suspended at one of its vertices, and a weight equal to half that of the plate is suspended from an adjacent vertex of the square. Find the point where the vertical through the point of suspension cuts the opposite diagonal of the square.

4. If three uniform rods be rigidly united so as to form half of a regular hexagon, prove that if the system is suspended from one of the triangles, one of the rods will be horizontal.

5. Prove that the straight lines which join the middle points of the opposite edges of a tetrahedron meet at a point which bisects each straight line.

6. Prove that if 2α be the angle of a sector whose C. G. lies on the chord of the arc, then $\tan \alpha = \frac{3}{2}\alpha$.

7. A solid right circular cone has its base scooped out so that the hollow is a right cone on the same base; how much must be scooped out so that the centre of gravity of the remainder may coincide with the vertex of the hollow?

8. Explain how to find the centre of gravity of the segment of a circle greater than a semi-circle.

9. Any point P is taken upon the diameter AB of a semicircular area, and the semi-circles upon AP, BP as diameters are removed. Find the centre of gravity of the area that is left. *(P. U. 1939)*

10. A circular tray of radius a stands on a single circular foot, of radius b. If P is the whole weight of the tray and its support, find how far from the centre a weight Q can be placed without the tray falling over. *(P. U. 1938)*

11. The tangent of the semi-vertical angle of a cone is $\frac{1}{\sqrt{2}}$. Show that its centre of gravity coincides with the centre of the circum-sphere.

12. Perpendiculars are drawn from the angular points A, B, C of a triangle to the opposite sides a, b, c, and another triangle is formed by joining the feet of these perpendiculars. If x, y, z be the distances of the C. G. of this triangle from the sides a, b, c, ; prove that

$$\frac{x}{a^2\cos(B-C)} = \frac{y}{b^2\text{co[illegible]}-A)} = \frac{z}{c^2\cos(A-B)}.$$

[*P. U. 1906*].

13. ABC is a triangle inscribed in a circle whose centre is O, and G_1, G_2, G_3 are the centres of gravity of the sectors AOB, BOC, COA respectively, prove that

$$\frac{AB}{OG_1}+\frac{BC}{OG_2}+\frac{CA}{OG_3}=3\pi.$$

14. If the centre of gravity of a tetrahedron be the common vertex of four tetrahedra whose bases are the faces of the original tetrahedron, show that the four tetrahedra are of equal volume.

15. A piece of wire is formed into a triangle ; find the distances of the centre of gravity from each of the sides and show that if x, y, z be the three distances and r the radius of the in-circle, then

$$4xyz-r^2(x+y+z)=r^3.$$

16. Show that the locus of the centres of gravity of all triangles inscribed in a circle having a common vertex at a fixed point and bases of constant length, is another circle.

17. Prove the following construction for *the centre of gravity of any trapezium* :

Let ABCD be the trapezium, AB ||CD.

Produce AB to L, BA to L′, CD to M and DC to M′. Make BL=DC, AL′=CD, DM=BA, CM′=AB. Then LM and L′M′ intersect in G.

18. Prive the folllowing construction for the *centre of gravity of any quadrilateral.*

Let O be the point of intersection of the diagonals AC, BD and M the middle of the line joining their middle points. Join OM and produce it to G making $MG=\frac{1}{3}$ OM. G is the C. G [*P.U, 1938*]

Or. Take CE=OA. Then the C. G. of △ BDE is the C. G of the quadrilateral ABCD.

19. If A and B be the positions of two masses m and n, and if G be their centre of mass, show that (P being any point) $mPA^2+nPB^2=mGA^2+nGB^2+(m+n)PG^2$.

If m_1, m_2......m_r be the masses placed at points A_1, A_2......A_r and G be their centroid, prove that

$$\sum_{k=1}^{r} m_k PA_k^2 = \sum_{k=1}^{r} m_k GA_k^2 + \sum_{1}^{r} \epsilon m_k . PG^2.$$

(P being any point).

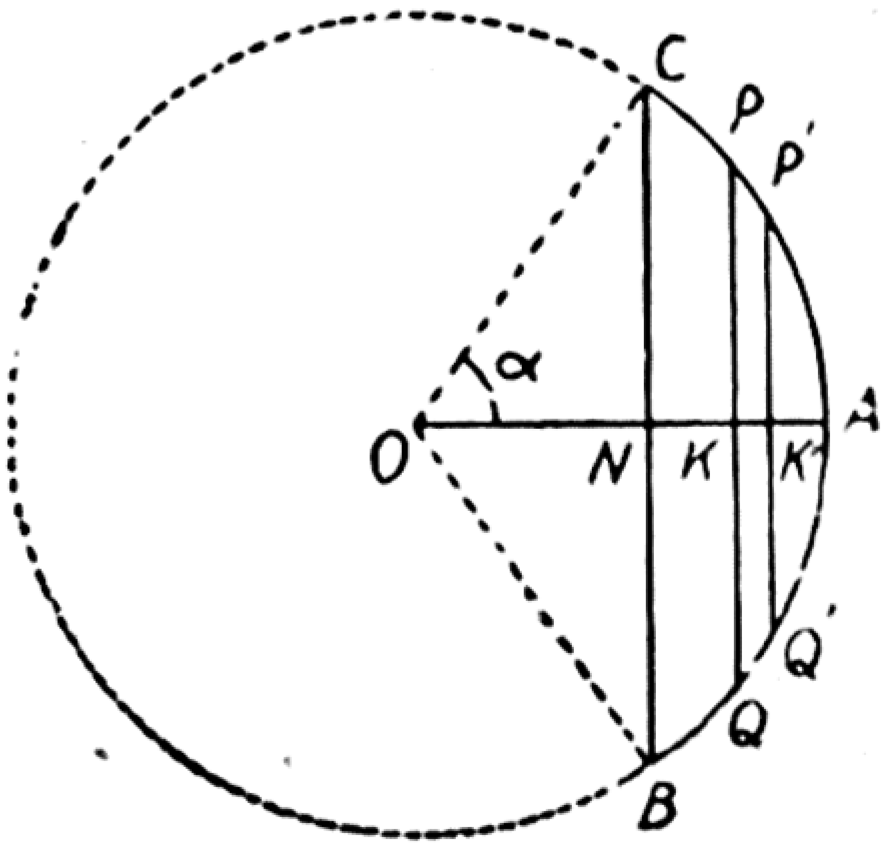
20. If two intersecting tangents are drawn from the extremities of a quadrant of a circle 4 feet diameter, find the distance of the C. G. of the area enclosed between the tangents and the arc from either tangent [*P. U. 1914*]

101. *To find the centre of gravity of a* **segment of a sphere.**

The perpendicular ONA from the centre O to the plane of the segment is an axis of symmetry. Take O as origin and OA as x-axis, and let a be the radius of the sphere.

Divide the segment into slices or discs parallel to its plane.

Consider any disc at distance $OK = x$. The volume of the disc $= \pi(a^2 - x^2)\delta x$ and its C. G. is in KK′.



Thus the centre of gravity of the segment with base of radius CN at distance $ON = r$ is given by the equation.

$$OG = \bar{x} = \frac{\int_r^a \pi(a^2 - x^2)dx.x}{\int_r^a \pi(a^2 - x^2)dx}$$

$$= \frac{\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_r^a}{\left[a^2x - \frac{x^3}{3}\right]_r^a}$$

$$= \frac{\frac{a^4}{4} - \frac{a^2r^2}{2} + \frac{r^4}{4}}{\frac{2a^3}{3} - a^2r + \frac{r^3}{3}} = \frac{3}{4} \quad \frac{(a^2 - r^2)^2}{(a-r)^2(2a+r)} = \frac{3}{4} \quad \frac{(a+r)^2}{2a+r}.$$

For a **hemisphere** $r=0$ and $\therefore$ OG$=\frac{3a}{8}$.

102. *To find the centre of gravity of a* **sector of a sphere.**

Let OBAC be the sector and G its C. G. (figure of last article).

The sector may be regarded as composed of the right cone of height ON ($=r=a\cos\alpha$) on the circular arc whose radius is BN$=a\sin\alpha$, and the segment BACNB.

Now the volume of the cone

$$=\tfrac{1}{3}\pi.\text{ON}.\text{BN}^2=\tfrac{1}{3}\pi\, a^3\cos\alpha\,\sin^2\alpha.$$

Now the volume of the segment

$$=\frac{\pi}{3}(2a^3-3a^2r+r^3)=\frac{\pi}{3}a^3(2-3\cos\alpha+\cos^3\alpha)$$

[See last Art.]

Thus G the centre of gravity of the sector is given by

$$\text{OG}\times[\tfrac{1}{3}\pi a^3\cos\alpha\,\sin^3\alpha+\frac{\pi}{3}a^3(2-3\cos\alpha+\cos^3\alpha)]$$

$$=\tfrac{3}{4}\,\text{ON}.\,\tfrac{1}{3}\pi a^3\cos\alpha\,\sin^2\alpha+$$

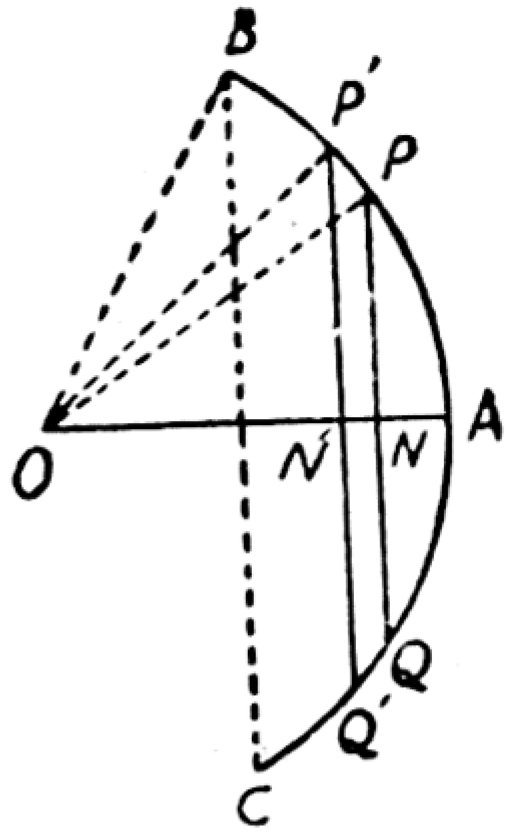
$$\tfrac{3}{4}\,\frac{(a+r)^2}{2a+r}\cdot\frac{\pi}{3}a^3(-3\cos\alpha+\cos^3\alpha).$$

$$\therefore\ \text{OG}=\tfrac{3}{4}a\,\frac{\cos^2\alpha\,\sin^2\alpha+\dfrac{(1-\cos\alpha)^2}{2+\cos\alpha}[2-3\cos\alpha+\cos^3\alpha]}{\cos\alpha\,\sin^2\alpha+2-3\cos\alpha+\cos^3\alpha}$$

$$=\tfrac{3}{4}a\,\frac{\cos^2\alpha\,\sin^2\alpha+(1+\cos\alpha)^2(1-2\cos\alpha+\cos^2\alpha).}{2(1-\cos\alpha)}$$

$$=\tfrac{3}{4}a\,\frac{\cos^2\alpha\,\sin^2\alpha+\sin^4\alpha}{2(1-\cos\alpha)}=\tfrac{3}{8}a(1+\cos\alpha)$$

103. *To find the centre of gravity of a* **spherical shell.**



Let O be the centre of the spherical shell OBC and a the radius of the sphere.

Let OA be the axis of symmetry.

Take sections of the shell by any two consecutive planes perpendicular to OA. The sections are circles having their centres at N, N′, the intersections of OA with the planes. Consider the ring cut off by these planes.

If $\angle AOP = \theta$, the width of the ring $= PP' = a\delta\theta$, and its circumference is in the limit, equal to the circumference of the circle PNQ. Thus the area of the ring $= 2\pi a^3 \sin\theta \, \delta\theta$, having its C. G. at N.

If G be the C. G. of the shell,

$$OG = \frac{\int_0^\alpha 2\pi\, a^2 \sin\theta \, d\theta . a\cos\theta}{\int_0^\alpha 2\pi a^3 \sin\theta \, d\theta} = \frac{a}{2} \frac{\int_0^\alpha \sin 2\theta \, d\theta}{\int_0^\alpha \sin\theta \, d\theta}$$

$$= \frac{a}{4} \frac{1 - \cos 2\alpha}{1 - \cos\alpha} = \frac{a}{2}(1 + \cos\alpha).$$

For a **hemi-spherical shell**, $\alpha = \frac{\pi}{2}$ and $\therefore$ $OG = \frac{a}{2}$.

Properties of the centre of gravity.

104. We have already shown that *if a heavy body is supported at its centre of gravity, it will balance in every position.* The following are obvious consequences of this proposition.

(i) *If a body is supported at a point or suspended from a point round which it can move freely, it will not rest unless its centre of gravity and the point of suspension, or support, are in the same vertical line.*

(ii) *If a body can revolve round an axis which is not vertical, it will not rest unless its centre of gravity is in the vertical plane passing through the axis.*

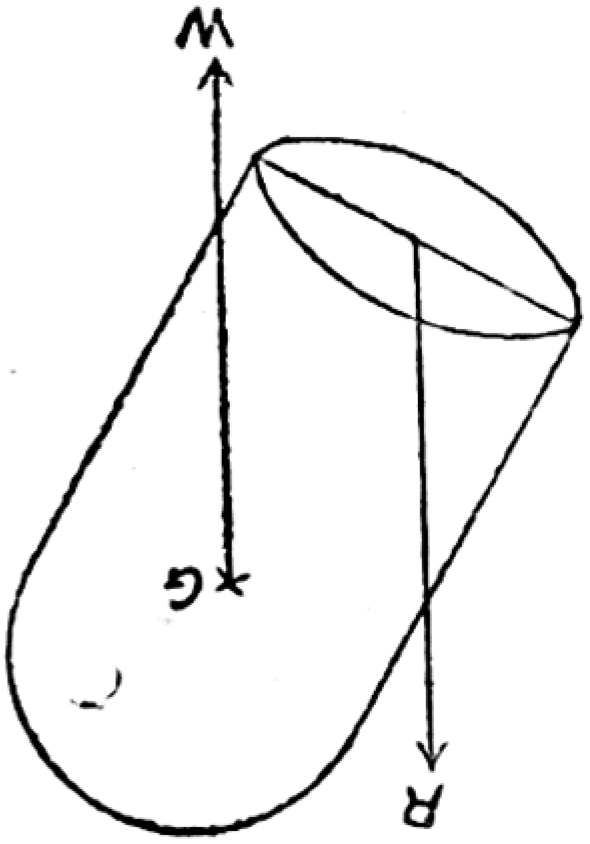
For, the body is acted upon by its own weight and the reactions of the axis. For equilibrium, the moment of the weight about the axis must vanish. This is possible only if the line of action of the weight is parallel to the axis or intersects it. But the axis is not vertical, hence the C.G. must be in the vertical plane through the axis.

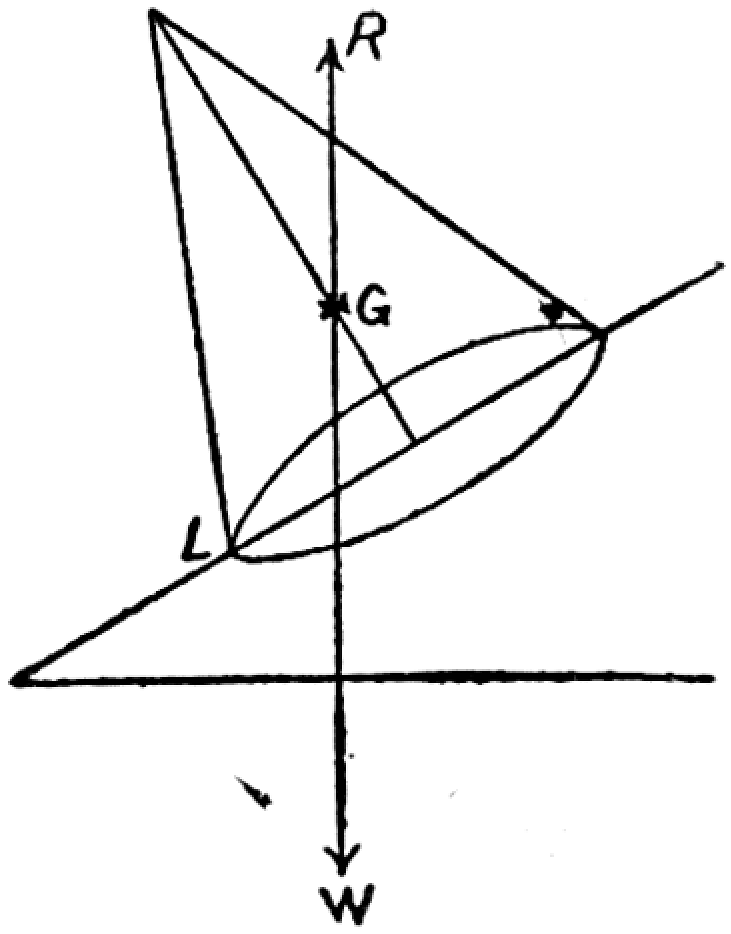
(iii) *When a body is placed on a plane, it will stand or fall according as the vertical line through its* C.G. *falls within or without the base on which it rests.*

N.B.—By base is here to be understood the area enclosed by stretching a string round the geometrical base, *i.e.* round the points of contact of the body with the plane.

(*If the plane is inclined,* it should be so *rough* as to prevent slipping).

Since the reactions of the points of contact of the body with the plane all act away perpendicularly from the plane and, therefore, form a system of like parallel forces, the line of action of their resultant cannot cut the plane outside the base of the body, for the resultant of two like parallel forces acts always at a point between the forces.



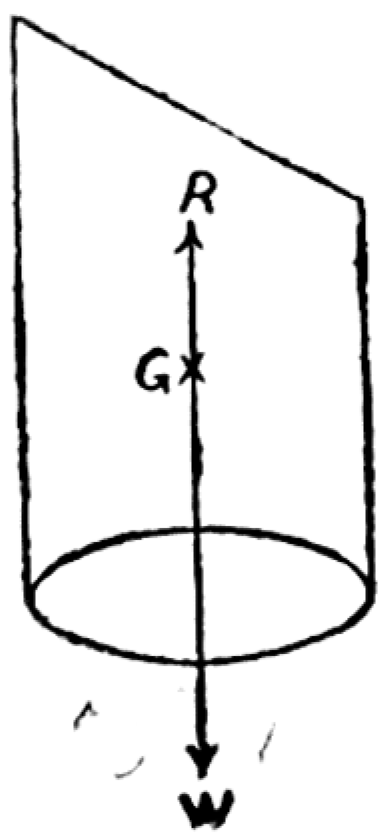


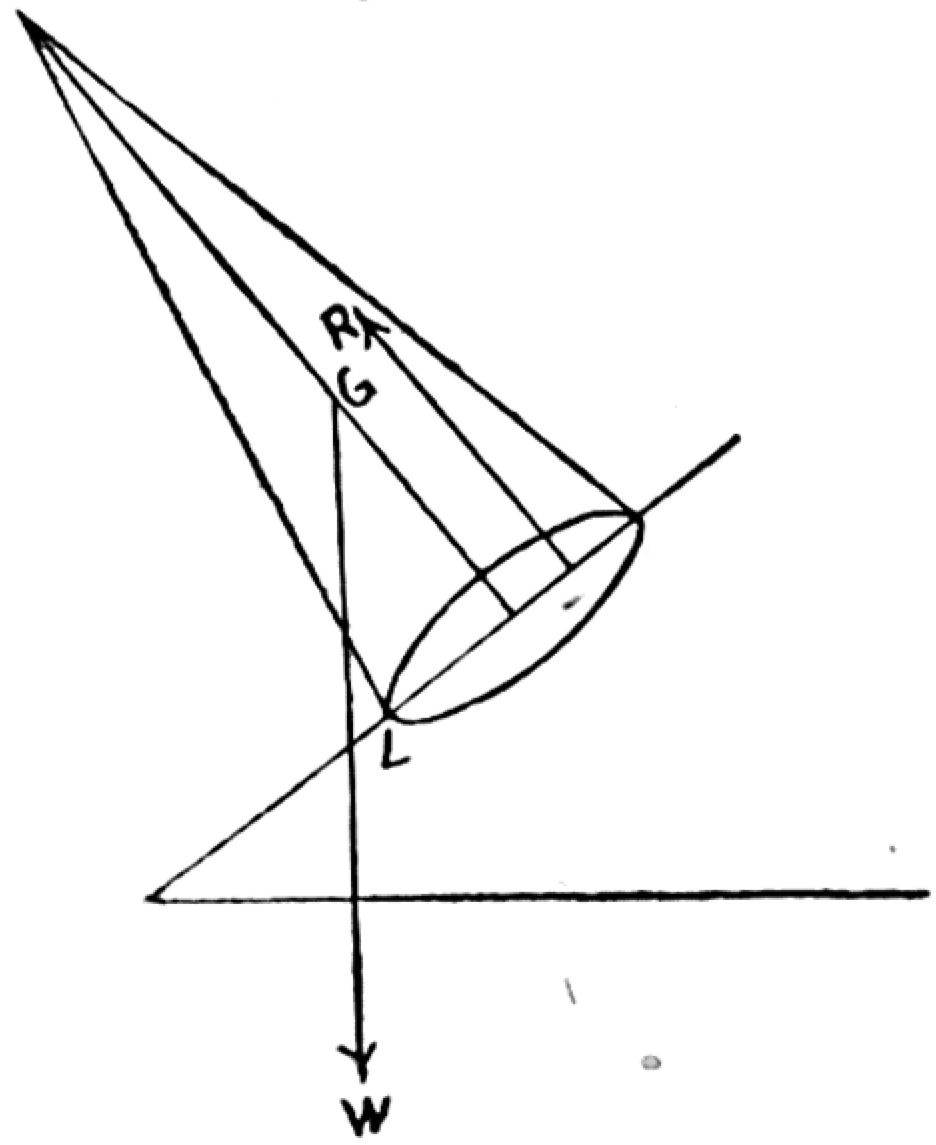
Now *firstly,* let the vertical through the C.G. of the body cut the supporting plane at a point inside the base of the body. Then the moment of the weight about any point on the edge of the base tends to press down other points of the body touching the plane, while the resultant reaction or the plane prevents the body from penetrating it, for reaction is a force which tends to resist or change motion, *i.e.*, has no tendency to produce motion but has a tendency to modify it.

Thus the resultant reaction will adjust itself so as to counteract the forces of gravity on the body as long as the vertical line through the C.G. falls inside the base.

Hence the dody will stand. And if it be slightly tilted up, its weight will bring it back to its original position as soom as it is let go.

Secondly, let this vertical through the C.G. cut the plane at a point K outside the base. Then the moment of the weight about a point L on the edge of the base nearest to K tends to turn the body about L so as to lift up all other points of the body touching the plane, for the resultant reaction





which acts away from the plane does not prevent any part of the body from being lifted off. Hence the body will fall.

If a body were to topple over, it would do so about some point or line in the plane. The moment of the weight about the point or the line tends to overturn the body and the moment of the resultant reaction about the same point resists this tendency. If the latter is greater or equal to the former, the body will remain in equilibrium, otherwise it will topple over.

Illustrations. A porter carrying a heavy trunk in one hand often extends the opposite arm at full length in order to bring more readily his C.G. over a point between his two feet.

A cart will overturn if the vertical through its C. G. falls outside the wheel base.

A man carrying a heavy weight in front of him leans back in order to bring his C.G. over his base.

A man leaning too far back in a chair tilted up on its hind legs shall fall over backwards when the vertical through his C.G. falls behind the line joining the two back feet.

(*Tutorial Statics.*)

105. Stable, unstable and neutral equilibrium. When a body is slightly displaced from a position of equilibrium, it may (*i*) tend to return to its original position, (*ii*) tend to recede from the original position, (*iii*) remain in equilibrium in the new position.

The original position of equilibrium is said to be *stable* in the 1st case, *unstable* in the 2nd case, and *neutral* in the third case.

The beam of a balance, a stick standing upright on the finger, and a door free to swing about a vertical line of the hinges, are examples of the three kinds of equilibrium.

A cone affords a good illustration of all three kinds of equilibrium. A cone resting on its base in contact with a horizontal plane is in stable equilibrium. A cone resting with its vertex in contact with the plane is in unstable equilibrium. A cone resting with its slant side in contact with the plane is in neutral equilibrium.

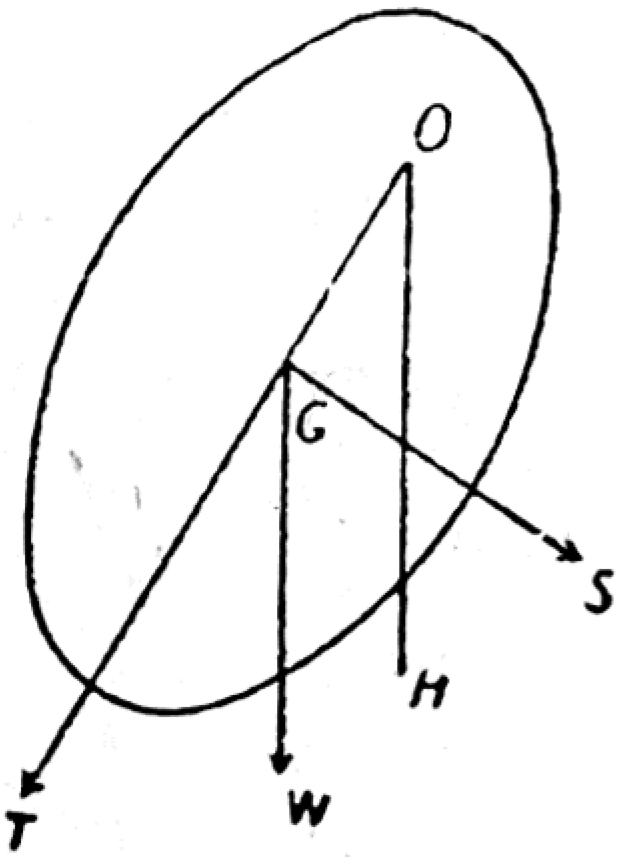
A heavy sphere resting on a horizontal plane is in stable, unstable or neutral equilibrium according as its centre of gravity is below, above, or at its centre.

106. When a heavy body, which can turn freely round a fixed point (or axis), is in equilibrium, *the equilibrium is stable, unstable, or neutral according as its centre of gravity is vertically below, vertically above, or coincident with* (lies on) *the fixed point* (or axis).

Let O be the fixed point, G the centre of gravity of the body, OH the vertical line through O.

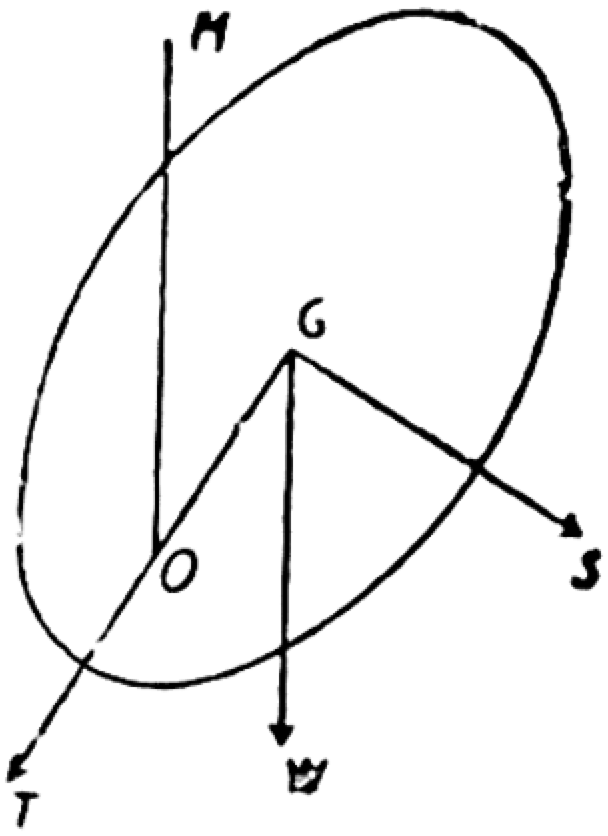
(*i*) *When G is on OH below O, the body is in stable equilibrium.*

For, suppose the body to be slightly displaced from the position of equilibrium, as in the figure.



Let the weight W of the body, acting vertically downwards through G, be resolved into two components T along OG and S perpendicular to OG. The moment of T about O is zero.

$\therefore$ S tends to turn the body round to its original position of equilibrium, with G vertically below O.

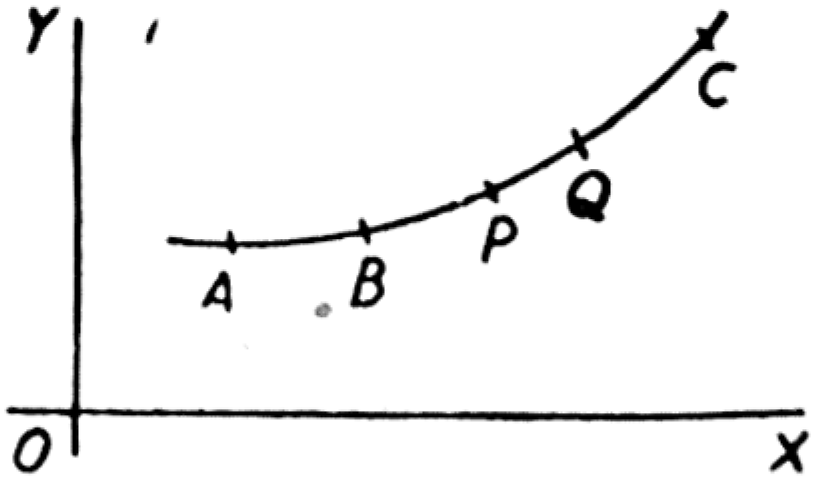


(*ii*) Similarly, it may be shown that *when G is vertically above O* on OH, S tends to turn the body away from its original position and the equilibrium is therefore *unstable.*

(*iii*) *When O coincides with G,* the body will remain in equilibrium in all positions and the equilibrium is, therefore, *neutral.*

Centres of Gravity of arcs and areas of curves.

***107.** *To find the centre of gravity of a* **given arc** *of the curve* $y=f(x)$.



Let ABC be the graph of the curve $y=f(x)$, when OX and OY are the co-ordinate axes.

The centre of gravity of arc BC is required.

Let $AP=s$, length of the arc being measured from some fixed point A, $PQ=\delta s$; x, y the co-ordinates of P.

Let k denote the area of a transverse section and ρ the uniform density. Then the mass of the element $PQ=k\rho\delta s$ and the co-ordinates of its centre of gravity are ultimately x, y.

$$\text{Hence } \bar{x} = \frac{\Sigma\rho k,\delta s.x}{\Sigma k\rho.\delta s} = \frac{\int x\,ds}{\int ds}, \text{ [if } k \text{ is constant].}$$

$$\bar{y} = \frac{\Sigma\rho k\delta s\, y}{\Sigma\rho k\delta s.} = \frac{\int y\,ds}{\int ds},$$

the integrals being taken between proper limits.

Since $\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$, we may write the formu
in the form :

$$\bar{x} = \frac{\int x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}{\int\sqrt{1+\left(\frac{dy^2}{dx}\right)}\,dx},$$

$$\overline{y}=\frac{\int y\sqrt{1+\left(\frac{dy^2}{dx}\right)}dx}{\int\sqrt{1+\left(\frac{dy^2}{dx}\right)}dx}.$$

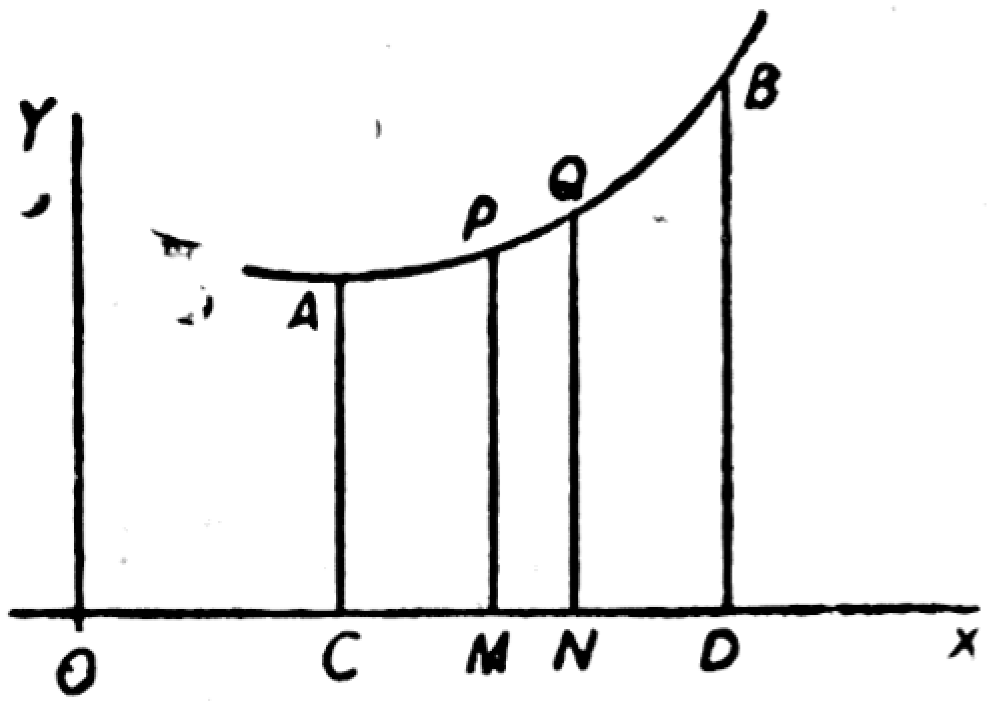
In polar co-ordinates,

$$w\frac{dr}{d\theta}=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}.$$

$$\therefore\ \overline{x}=\frac{\int r\cos\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta}{\int\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta},$$

$$\overline{y}=\frac{\int r\sin\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta}{\int\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta}.$$

108. *To find the centre of gravity of* **an area bounded by the curve y=*f* (x), two ordinates, and x-axis.**



Let ABDC be an area bounded by the curve AB, ordinates CA and DB and the portion CD of the x-axis.

Divide CD into n parts and draw ordinates at the points of division.

Let MP, NQ be two adjacent ordinates, where P is (x, y), MN$=\delta x$.

The area of the strip PQNM$=y\delta x\times\epsilon$ and the co-ordinates of its centre of gravity are (x', y'), where

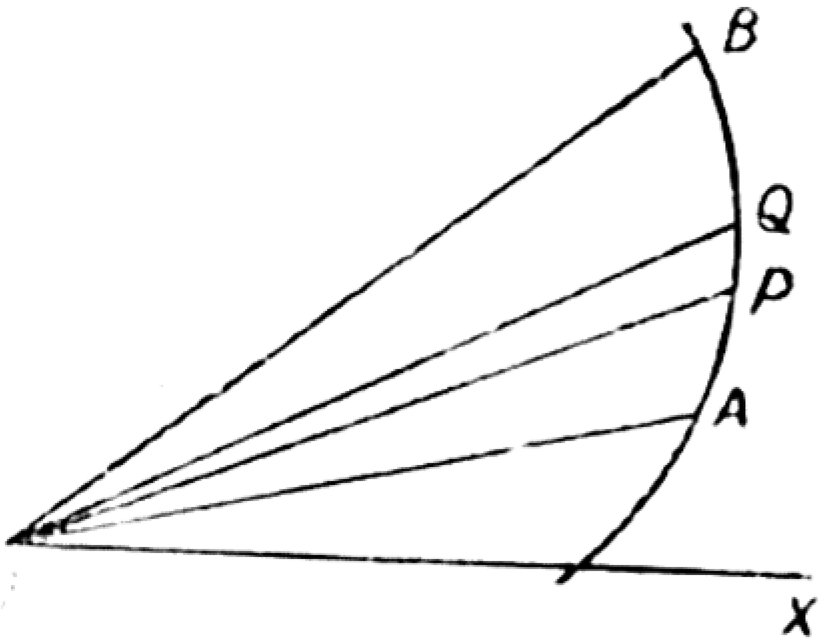
$\epsilon\to 0$, $x'\to x$, $y'\to\frac{1}{2}y$ when $n\to\delta$.

Thus the co-ordinates of the centre of gravity of the area are given by the formulæ :

$$\overline{x}=\frac{\Sigma k\rho(y\delta x+\epsilon)x'}{\Sigma k\rho(y\delta x+\epsilon)}=\frac{\int xy\ yx}{\int y\ dx},$$

$$\bar{y} = \frac{\Sigma k\rho(y\delta x+\epsilon)y'}{\Sigma k\rho(y\delta x+\epsilon)} = \frac{\frac{1}{2}\int y^2 dx}{\int y\, dx}.$$

where k and ρ are the uniform thickness and density of the area.



If the curve be given by the equation $r=f(\theta)$, divide $\angle$AOB into n parts, of which $\angle$POQ is one.

Let $\angle XOP=\theta$, $POQ=\delta\theta$. The abscissa and ordinate of the C. G. of area POQ $(=\frac{1}{2}r\delta\theta)$ are $\frac{2}{3}r\cos\theta$, $\frac{2}{3}r\sin\theta$.

$$\therefore\ \bar{x} = \frac{\int \frac{2}{3}r\cos\theta\frac{1}{2}r^2 d\theta}{\int \frac{1}{2}r^2 d\theta},$$

$$\bar{y} = \frac{\int \frac{2}{3}r\sin\theta\frac{1}{2}r^2 d\theta}{\int \frac{1}{2}r^2 d\theta}.$$

(Integrals being taken between the proper limits.)

109. *To find the centre of gravity of* **a surface or a solid of revolution generated by a curve $y=f(x)$.**

The axis of revolution is evidently an axis of symmetry. Take it as x-axis.

(*i*) Let the arc BC of the curve revolve about the x=axis (See figure Art. 106.)

The mass of the element of the shell generated by the elementary arc $PQ=2\pi k\rho y.\delta s$, where k and ρ are the uniform thickness and density. Thus for a surface of revolution,

$$\bar{x} = \frac{\int 2\pi k\rho y ds.x}{\int 2\pi k\rho y ds} = \frac{\int yx\, dx}{\int y\, ds},$$

Since $\dfrac{ds}{dx} = \sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$

$$\therefore \quad \bar{x} = \frac{\int yx\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}{\int y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx},$$

the integrals being taken between proper limits.

(*ii*) Let the elementary area PQNM revolve about the x-axis. (Fig. Art. 108). Then the elementary mass of the solid$=k\rho 2\pi y^2 \delta x$, where k, ρ are the uniform thickness and density.

Thus for a solid of revolution,

$$\bar{x} = \frac{\int 2\pi\, k\rho xy^2 dx}{\int 2\pi\, k\rho y^3 dx} = \frac{\int xy^3 dx}{\int y^2 dx},$$

the integrals being taken between proper limits.

110. Theorems due to Pappus or Guldins.

(*a*) *If an arc of a plane curve revolve about an axis in its plane, not intersecting it, the area of the surface generated is equal to the length of the arc multiplied by the length of the path of its centre of gravity.*

(*b*) *If a plane area revolve about an axis in its plane, not intersecting it, the volume of the solid generated is equal to the area multiplied by the length of the path of its centre of gravity.*

Take the axis of revolution as x-axis.

(*a*) Let S be the area of the surface generated by a given arc of the curve $y=f(x)$.

$$S=2\pi \int yds,$$

the integration extending over the arc.

If $\bar{y}$ be the C. G. of the arc, then

$$\bar{x} = \frac{\int y\, ds}{\int ds}.$$

$\therefore$ $S = 2\pi y \times s$, where s is length of the whole arc.

(*b*) If ΔA be an element of the area, the volume V generated in a complete revolution :

$$V = 2\pi \int y dA.$$

But if $\bar{y}$ be the C. G. of the generating area,

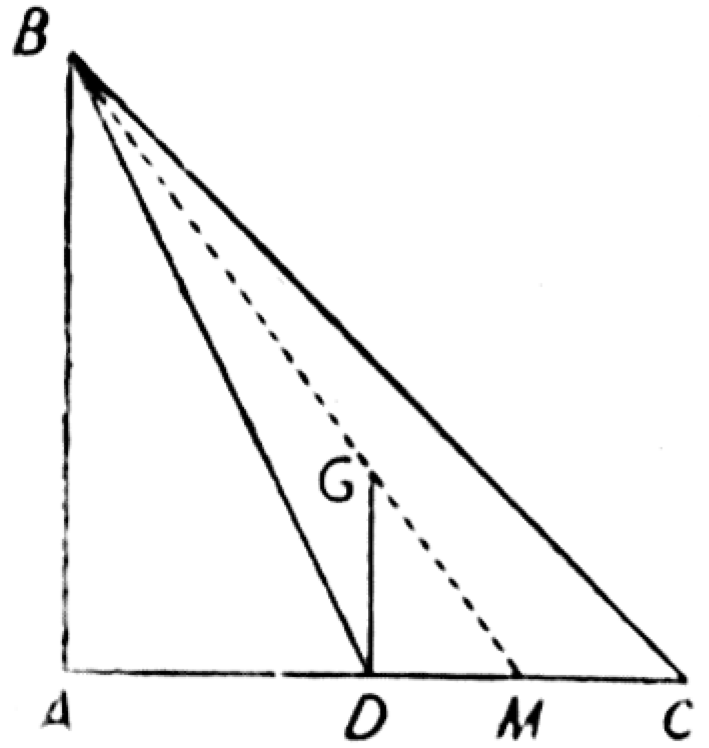
$$y = \frac{\int y dA}{\int dA}. \qquad \therefore\ V = 2\pi \bar{y} \times A.$$

We need not, however, take complete revolutions in the above.

These theorems may be used to find the C. G. of a plane arc or of a plane area when the surface or the volume generated by its revolution is independently known.

WORKED OUT EXAMPLES

111. Ex. 1. *ABC is a flat board, A being a right angle, and AC in contact with a flat table. D is the mid-point of AC and ΔABD is cut away. Show that the triangle BDC is just on the point of falling over.*



Let BM be the median of the ΔBDC. Through D draw DG$\parallel$AB, cutting BM in G.

Then since GM : GB $=$ MD : DA $= 1 : 2$

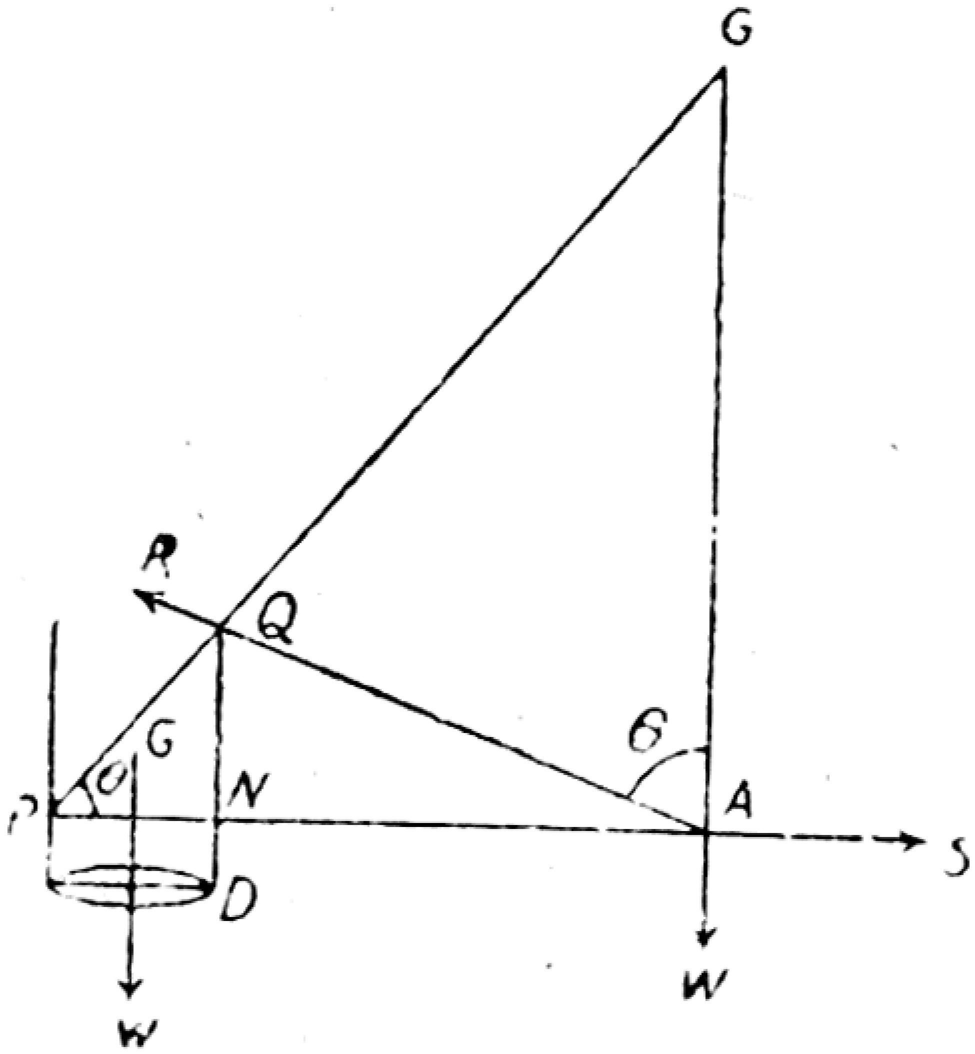
$\therefore$ G is the centre of gravity of Δ BDC.

Thus the line of action of the weight of the triangle BDC passes through the extreme point D of the base line CD.

Hence Δ BDC is just on the point of toppling over.

Ex. 2. *A uniform rod of length 32a rests partly within and partly without a smooth cylindrical cup of radius a. Show that, in the position of equilibrium the rod makes an angle of 60° with the horizon and prove also that the cylinder will topple over unless its weight be at least six times that of the rod.*

There are three forces acting on the rod—the reactions R, and S at the points Q and P, R acting perpendicularly to the rod and S acting horizontally; and its weight acting at G vertically downwards whose line of action must pass through A, the point of intersection of R and S. Let W and w be the weights of the rod and the cup.



From the equilibrium of the rod, we have

$$\begin{aligned} PQ &= AP \cos\theta \\ &= PG \cos\theta . \cos\theta \\ &= 16a \cos^2\theta . \end{aligned}$$

Also $PQ = PN \sec\theta = 2a \sec\theta$.

$\therefore \cos^3\theta = \frac{1}{8}$, *i.e.*, $\cos\theta = \frac{1}{2}$.

i.e., $\theta = 60°$.

Let D be the point of contact of the base of the cup with the ground which is nearest to the line of action of the weight.

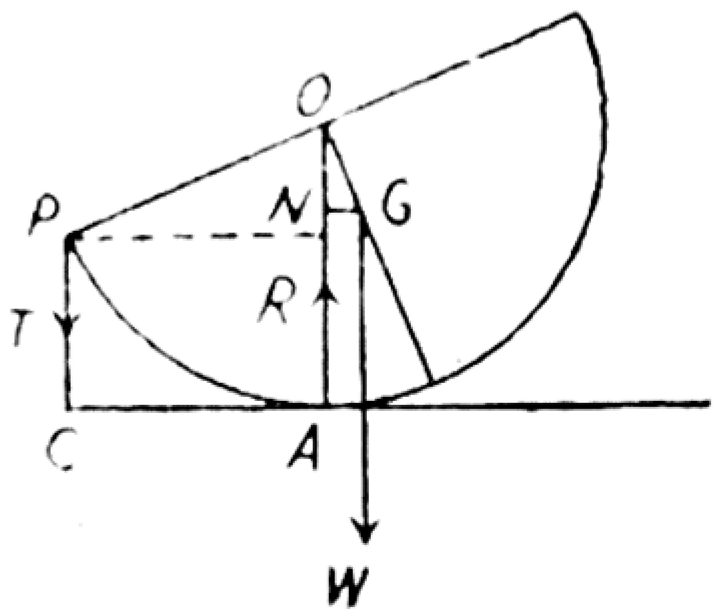
The cup will not topple over, if the moment of the weight of the rod about D is not greater than the moment of the weight of the cylinder about D, *i.e.*, if

$$\begin{array}{ll} W \times AN \ngtr w \times a & \left[\begin{array}{l} \quad AN = 6a, \\ \because \; AP = 16a \cos 60° = 8a. \end{array} \right. \\ i.e., \text{ if } w \nless 6W. & \end{array}$$

Ex. 3. *A hemisphere of radius a and weight W is placed with its curved surface on a smooth table and a string of length $l (l < a)$ is attached to a point on the rim and to a point on the table. Prove that the tension of the string is*

$$\frac{3}{8} W \frac{a - l}{\sqrt{2al - l^2}} .$$

Let A be the point of contact of the hemisphere (centre O) with the table, PC the string, P being on the rim and C on the table.



The hemisphere is acted upon by three forces—its weight W acting at G, its C. G., reaction R of the table on the hemisphere and the tension T of the string. These three forces must meet at a point, or must be parallel. Since W and R are parallel, all the three forces are, therefore, parallel.

Taking moments about A, we have

T.AC$=$W.GN, where GN$\perp$OA. ...(1)

Now CA$=$perpendicular from P to OA

$$=\sqrt{\text{OP}-(a-l)^3}=\sqrt{a^2-(a-l)^2}$$

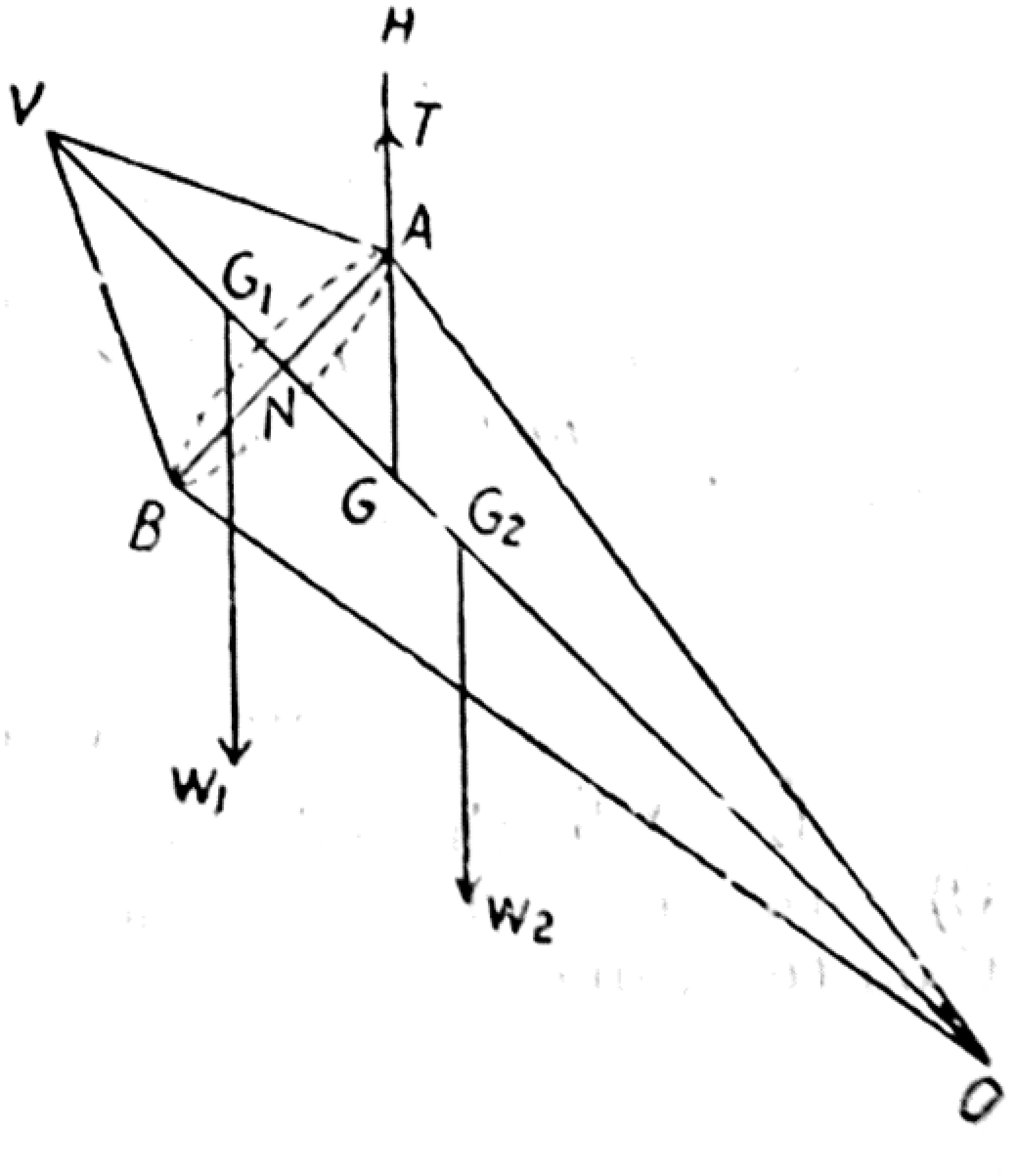
$$=\sqrt{2al-l^2}.$$

$$\text{Sin GON}=\cos \text{POA}=\frac{a-l}{a}, \text{ and OG}=\tfrac{3}{8}a.$$

$$\therefore \text{(1) gives T.}\ \sqrt{2al-l^2}=\text{W}.\tfrac{3}{8}a.\ \frac{a-l}{a}.$$

$$\therefore\quad \text{T}=\tfrac{3}{8}\,\text{W}\frac{a-l}{\sqrt{2al-l^2}}.$$

Ex. 4. *A spindle-shaped solid of uniform material is bounded by two right circular cones of altitudes 6 and 2 inches with a common circular base of radius 1 inch. It is suspended by a string attached to a point on the rim of the circular base. Find the inclination of the axis of the spindle to the vertical, when it is hanging freely.*



Let the spindle formed by the cones OAB, VAB be suspended from A, and let N be the centre of the common base; G_1, G_2, the centres of gravity of the cones, where $NG_1=\frac{1}{2}$, $NG_2=\frac{3}{2}$. The spindle is acted upon by three forces, weight of the cone VAB, weight of the cone OAB, and the tension of the string.

These three forces are evidently parallel. The line of action of the weights divides G_1G_2 at G in the inverse ratio of their weights.

$\therefore\quad 2G_1G=6GG_2, \qquad i.e.,\quad G_1G=3GG_2.$

Now $G_1G_2=G_1N+NG_2 \qquad =\frac{1}{2}+\frac{3}{2}=2.$

$\therefore\quad G_1G=\frac{3}{2}. \qquad$ Thus $NG=1=AN$.

For equilibrium, G is vertically below A.

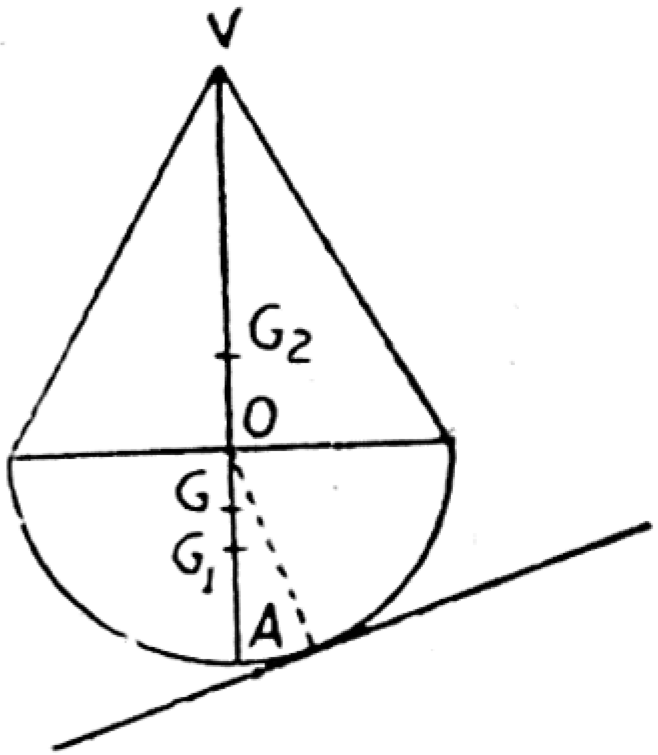
$\therefore$ the axis is inclined to the vertical at 45°.

Ex. 5. *If a cone have its base united concentrically to the base of a hemisphere of equal radius r, find the height of the cone that the solid may rest on a horizontal table at any point of its spherical surface.*

Let G_1 and G_2 be the centres of gravity of the hemisphere and the cone, and let A be the point of contact which is initially in contact with the table. Let O be the centre of the common base. If h be the height of the cone, $OG_2=\frac{1}{4}h$, $OG_1=\frac{3}{8}r$.

Also the weights of the hemisphere and cone are proportional to $\frac{2}{3}r^3$ and $\frac{1}{3}r^3h$.

The reaction of the table, in the displaced position of the body, always passes through O.



The equilibrium is stable, unstable, or neutral according as G, the centre of gravity of the body, is below, above, or at O ;

$i.\ e.$ according as $OG_1\times\frac{2}{3}r^3 \overset{>}{\leqslant} OG_2\times\frac{1}{3}r^2h,$

$i.\ e.$ according as $\dfrac{3r}{8}\times\frac{2}{3}r^3 \overset{>}{\leqslant} \frac{1}{4}h\times\frac{1}{3}r^2h,$

$i.\ e.$ according as $3r^2 \overset{>}{\leqslant} h^2.$

Thus for neutral equilibrium, $h=\sqrt{3}\,r.$

Exercises XIII

1. If a triangular lamina can just rest in a vertical plane with its edge AB in contact with a smooth table, prove that

$$BC^2 \sim AC^2 = 3AB^2.$$

2. Explain why in ascending a hill we appear to lean forwards ; in descending, to lean backwards.

3. Why does a person rising from a chair bend his body forwards and his leg backwards ?

4. Prove that the pull exerted by a man in a tug of war is $\frac{a}{b}$ of his weight, where a is the horizontal projection of a line joining his heels to his centre of gravity, and b is the height of the rope above the ground.

5. What is the use of a rope dancer's pole ?

6. How would you test the nature of equilibrium of a body at rest ? Point out the advantages of three-legged and four-legged tables, respectively ?

7. A uniform rod AB of mass 1 lb. is 12 inches long, and is hinged at a point C, 3 inches from A. Masses of 6 lbs. and 1 lb. are fastened at A and B respectively, and the rod is placed with A vertically over C. (*i*) Is equilibrium possible ? (*ii*) If so, is it stable or unstable ? (*iii*) What extra weight fastened at B would cause neutral equilibrium ?

8. A cylinder of height 2 ft. and radius 1 ft. is placed on an inclined plane and prevented from sliding, if the inclination of the plane be gradually increased, find when the cylinder will topple.

9 A homogeneous body, consisting of a cylinder and a hemisphere joined at their bases, is placed with the hemispherical end on a horizontal table ; is the equilibrium stable or unstable ?

10. A cylindrical hole of radius a is drilled through a hemisphere of radius b so that the radius from perpendicular to the base of the hemisphere is also the central line of the hole. Find the centre of gravity of the figure.

11. A right circular cone whose base is of radius r is divided into two equal parts by a plane through the axis. Prove that the distance of the centre of gravity of either half from the axis is $\frac{r}{\pi}$.

12. A round table stands on three vertical legs placed on the circumference at equal distances. Prove that a weight less than that of the complete table may be placed anywhere on it without upsetting it.

13. Prove that the centre of gravity of the part of the surface of a sphere cut off by two parallel planes is midway between the planes on the axis of symmetry.

14. A solid is formed by a uniform solid cone of semi-vertical angle α and the major segment of a sphere of radius r, so that the generating lines of the cone are tangential to the sphere at the circle of contact. Prove that the distance of the centre of gravity of the solid from the centre of the sphere is $\frac{1}{4}r\,(1-\sin\alpha)^2/\sin a\,\alpha$.

15. If a cone be placed with its base in contact with a smooth inclined plane and be prevented from slipping by a horizontal force applied at the apex, give a diagram showing the arrangement of the forces acting on it.

Also if the inclination of the plane be 45° and the radius of the base be equal to ·375 of the height, prove that the cone will be on the point of toppling over.

16. Prove that a thin hemispherical bowl can rest in equilibrium with its curved surface in contact with a rough inclined plane if the inclination does not exceed 30°.

17. ABC is an isosceles triangle of weight 3 lbs. of which the angle=120°, and the side AB rests on a smooth horizontal plane, the plane of the triangle being vertical; if a weight of 1 lb. be hung at C, show that the triangle will just be on the point of toppling over.

18. A square board ABCD has a mass of 10 lbs. If it is suspended from the point A, what mass must be attached at B, in order that the board may rest with AB inclined to the horizontal at an angle of 60°.

19. A portion of uniform thin spherical shell is bounded by a circular rim. When the shell is freely suspended from a point in the rim, the radius through the lowest point of

the rim is horizontal. Show that the plane bounding the shell is at a distance from the centre of the sphere equal to two-thirds of the radius.

20. A solid sector of a sphere hangs from a point in its circular rim with its axis horizontal ; finds its vertical angle.

21. A hemispherical shell has its inner and outer radii a, b. Show that the distance of its centre of gravity from its centre along the axis of symmetry is $\frac{3}{8}\ \frac{(a+b)(a^2+b^2)}{(a^2+ab+b^2)}$.

Deduce the position of the C. G. of a thin hemi-spherical shell.

22. A frustum is cut from a right cone by a plane bisecting the axis and parallel to the base ; show that it will rest with its slant side on a horizontal table if the height of the cone bears to the diameter of its base a ratio greater than $\sqrt{7} : \sqrt{17}$.

23. Apply Guldin's theorem to find the C. G. of the area and also of the arc of a semi-circle.

24. Use Guldin's theorem to find the volume of the frustum of a right cone in terms of its altitude and the radii of its ends.

25. Find the centre of gravity of (*i*) an arc (*ii*) a sector of the circle $x^2+y^2=a^2$.

26. Find the C. G. of the area between two semi-circles having a common diameter and touching internally at a point.

27. Find the centre of gravity of (*i*) the surface of revolution (*ii*) the volume of revolution generated by the circle $x^2+y^2=a^2$ about any diameter.

28. Find the C. G. of the hemi-spherical shell and so generated by the semi-circle given by $x^2+y^2=a^2$.

REVISION PAPER III

1. A rod of length $5a$, is bent so as to form 5 sides of a regular hexagon. Show that the distance of its centre of gravity from either end of the rod is $\frac{a}{10}\sqrt{133}$. (*P.U.1932*)

2. From a uniform rectangular lamina, 10 inches by 8 inches, a portion, 6 inches square is cut out from one corner. Find the C. G. of the remainder. (*P .U. 1941*)

3. Three parallel forces P, Q, R, act, at the vertices A, B, C of a triangle, and are proportional respectively to a, b, c, the sides of the triangle. Show that their centre coincides with the incentre of the triangle. (*P.U. 1940*)

4. G is the centre of gravity of a uniform quadrilateral plate. G′ is the centre of gravity of four equal particles placed at its corners, and O is the intersection of its diagonals. Prove that O, G, and G′ are in the same st. line, and that $OG' = 3GG'$. (*P.U. 1938*)

5. O is the centre of a square sheet of paper ABCD, in which the length of AO is l, C is folded over on to O, and the sheet is creased, and lies flat. Show that the distance of the centre of mass from A is $\frac{23l}{48}$. (*P.U. 1940 Supp.*)

6. If A, B, and C be the positions of three masses m, n and p, and if G be their centre of gravity, show that if P be any point, then

$$m.AP^2 + n.BP^2 + p.CP^2 = m.AG^2 + n.BG^2 + p.CG^2 + (m+n+p)PG^2$$

(*P.U. 1933 Supp.*)

7. From a body of weight W, a weight w is cut off and moved a distance x. Show that the centre of gravity of the whole is there by moved a distance $\frac{xw}{W}$ in that direction.

(*P. U. 1935*)

8. A uniform lamina in the shape of a rt. angled triangle and such that one of the sides containing the right angle is three times the other, is suspended by a string attached to the right angle; in the position of equilibrium, show that the hypotenuse is inclined at angle of $\sin^{-1}\frac{3}{5}$ to the vertical.

(*D. U. 1937*)

• 9. Particles of weights 1, 2, 3, 4, 5, 6, lbs. respectively are arranged in this order round the circumference of a circle of radius 10 inches at intervals of 60°. Find the distance of their centroid from the centre of the circle. (*P. U. 1936*)

10. A cone, whose height is equal to four times the radius of its base, is hung from a point in the circumference of its base. Show that it will rest with its base and axis equally inclined to the vertical. (*P. U. 1937*)

11. From a sphere of radius R is removed a sphere of radius r, the distance between their centres being c; find the centre of the mass of the remainder.

12. A uniform rod of length l with its cross-section a circle, is sharpened at one end, like a pencil to a conical point without shortening the axis of the cylinder. If the height of this cone is h, show that the centre of gravity has been displaced by a distance $\frac{h}{4}\,\frac{(4l-3h)}{3l-2h}$ *(D. U. 1940)*

13. The centre of mass of an arc of a circle is at a distance $\frac{a\sin\alpha}{\alpha}$ from the centre, a being the radius of the circle of which the arc forms a part and 2α the angle subtended at the centre of the arc. Show that the centre of the mass of a segment of the circle is in the diametrer bisecting the segment, at a distance $\frac{4}{3}\,\frac{a\sin^3\alpha}{2a-\sin^2\alpha}$ from the centre. Hence obtain the centre of mass of a semi-circular arc. (*P. U. 1933*)

14. A thin uniform rod is bent at one end to form a walking stick with a semi-circular handle. The straight portion AB is of length $2l$, and the curved portion BC is of radius a (less than l). The straight line joining the ends B and C of the semi-circular portion is perpendicular to AB. The stick hangs freely with the end C supported on a horizontal table. Prove that the straight part makes an angle θ with the vertical where

$$\tan\theta=\frac{a(\pi a+4l)}{2(l^2-a^2)}. \qquad (P.\ U.\ 1941)$$

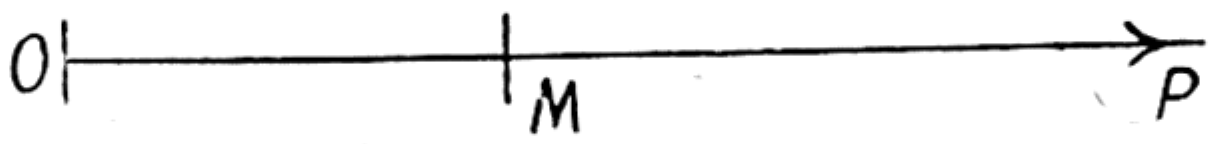
15. The altitude of a cone is h and the radius of its base is r; a string is fastened to the vertex and to a point on the circumference of the circular base and is then put over a smooth peg. Show that if the cone rest with its axis horizontal, the length of the string must be $(h^2+4r^2)^{\frac{1}{2}}$.

CHAPTER IX

WORK

112. Definition and measurement of Work. A force is said to be *doing work* when its point of application undergoes a displacement. If the force remains constant in direction and magnitude, the work done is measured by *the product of the force and the displacement on* **the direction of the force.**

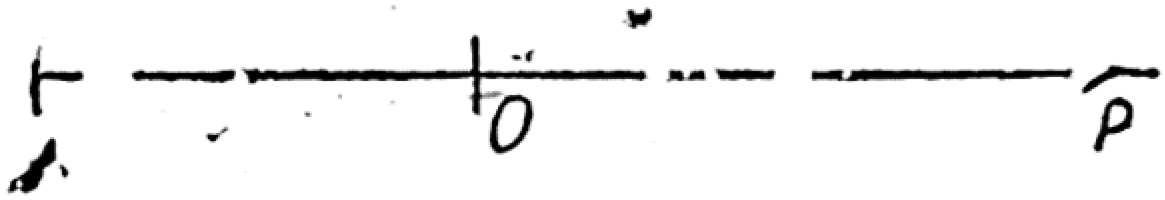
Let a body move under the action of forces so



that the point of application of the force P moves from O to M, then the amount of work done$=$P$\times$OM.

This work is positive since the displacement OM is in the direction of the force.

If the displacement is in a sense opposite to the direction of P, so that M is on the left of O, then the *work done is*

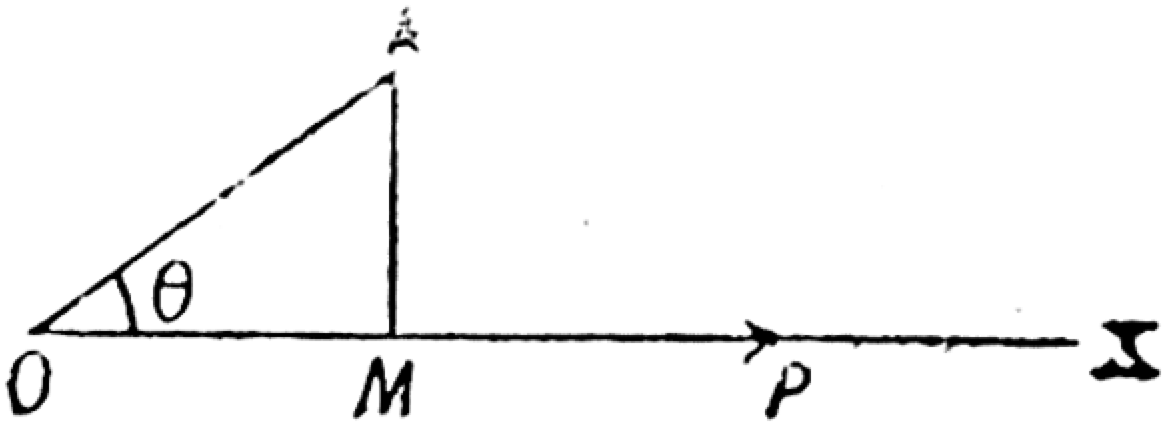


negative and is equal to P$\times$OM.

The student should *note* that there must be *other forces* acting on the body besides P. For, if P would have been the only force, the displaced position M could not have been in a direction opposite to that of P. It would, therefore, be more appropriate to say that the amount of work *contributed by* P$=$P$\times$OM.

113. Work done when the force is oblique to the displacement.

(*i*) Let a body move from O to A in a direction making an angle θ with the direction of the force P ; then OA *is the actual displacement.*

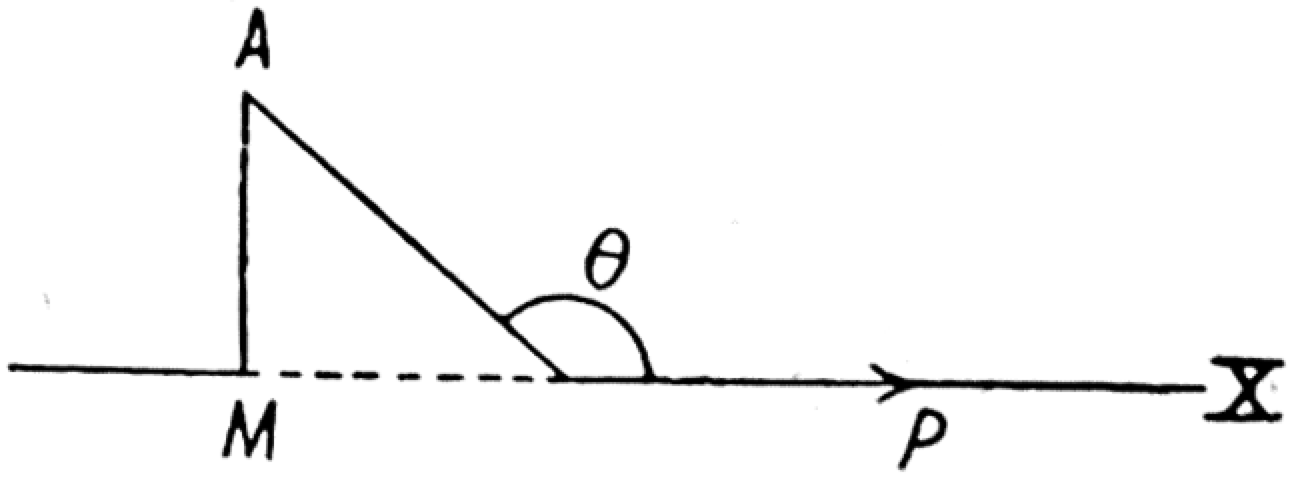


Draw AM perpendicular to the direction of P so that OM is *the projection of the displacement OA* on OX ; OM is described as *the displacement in the direction of P*. The work done by P is, then defined by the product P×OM=P. OA cos θ, which is positive *as θ is acute.* Thus,

work done=(force) × (displacement in the direction of force).

Note. Since work done=P×OA cosθ=P cos θ×OA.

∴ Work done is also =(*actual displacement*)
× (*resolved part of force in the direction of displacement*).



(*ii*) If M falls to the left of O, *the displacement in the direction of the force is negative* and *the work done by P will be negative*. In this case, θ is obtuse.

(*iii*) If the body moves *at right angles to the direction* of P, O and M will coincide and the *projection OM is zero.* The *work done will be nill.* (zero). For example, when a heavy body moves along a horizontal plane, *no work is done* **by gravity.**

Observation. A force, however large, does no work, if its point of application does not move. It should be observed that *both forces and motion are necessary in order that work may be done.* If we push a heavy stone resting on a rough horizontal plane but do not succeed in moving it, we do no work in *the technical* sense of the term even if we perspire profusely in the attempt. From the *popular point of view*, work is said to be done by a person whenever there is a sensation of fatigue in an activity even if it does not produce any motion, *but mathematical work requires both motion and force* for its production.

114. Unit of work. Work is the product of force and displacement ; and since the unit of displacement is one foot and the unit of force (in Statics) is the weight of one pound, the *unit of work is the work done by a force of one pound in moving its point of application through one foot in its own direction* and is called a **foot-pound**, *i.e.*, the unit of work is the work done by such a force P in such a displacement OA that the work P × OA is equal to unity. This is the *practical unit of work* in the British System, used by Engineers. In the French System, the unit of work is a *Centimetre-gramme*, which is the work done by a force of one gramme (weight of one gramme) in moving its point of application through one centimetre.

Note.—There are two kinds of unit of work, the *gravitational unit and the absolute unit*. In statics, the gravitational unit is always employed ; the absolute or *scientific unit of work is foot-poundal* $= \dfrac{\textit{foot-pound}}{32}$ in the British System, and centimetre-dyne (erg) $= \dfrac{\textit{centimetre-gramme}}{981}$ in the French System.

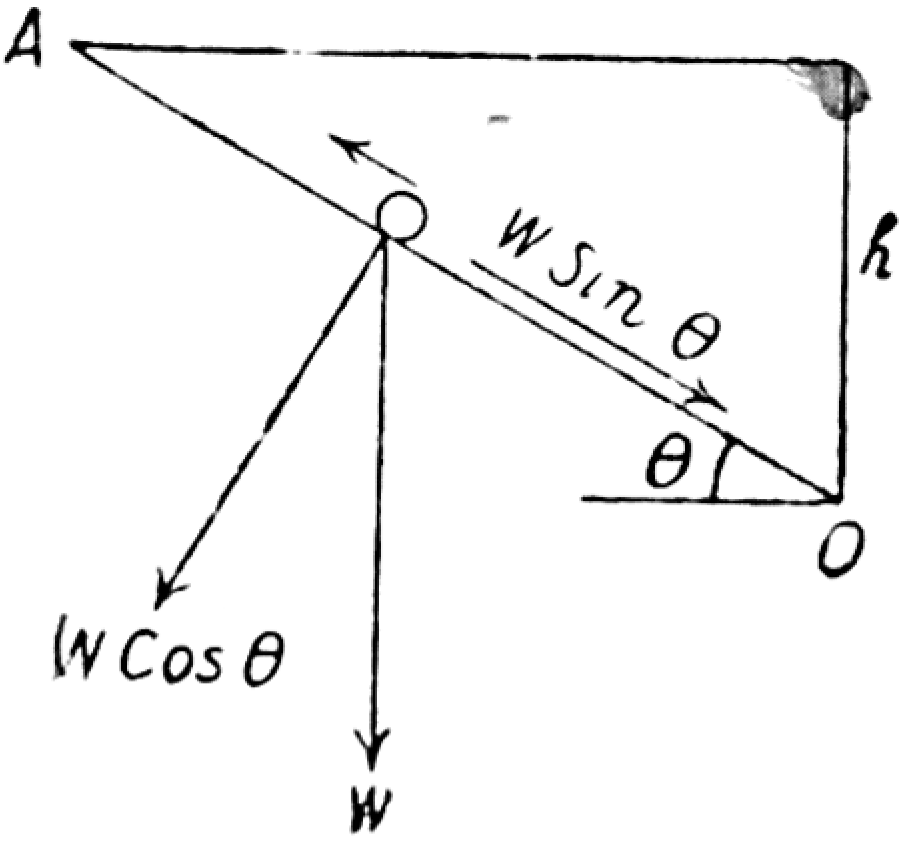
115. Positive and negative work. *The displacement OM (not MO) is positive* if it is in the same direction as the force P ; and *negative* if it is opposite in sense to the direction of P. It is thus, evident from what has been said above that the work done by a force (being equal to P×OM) *is positive if the displacement OM is positive, and negative if the displacement is negative. i. e.*, if a body of weight W pounds *falls down* vertically through h feet, the work done *by gravity is positive* and is equal to W h foot-pounds,and if the *body rises* vertically through h feet, *the work done by gravity is negative*, being equal to $(-Wh)$ *foot-pounds* of work.

Note.—When the work done by a force is *negative, the particle is said to do work* against the force so that *work done by a force = minus times work done by the particle.*

116. Work done against gravity in moving a body up along an inclined plane.

Let a body of weight W *move upward* through a distance l, along a plane inclined at an angle θ to the horizontal.

Revolving W along and perpendicular to the plane, we have W equivalent to two simultaneous forces W sin θ and W cos θ, as shown.



As the body moves from O to A, *the work done against W* cos θ is zero, since W cos θ is, at each instant, perpendicular to the direction of motion, Art. 114 (*iii*) ; the *work done against W sinθ* $= W\sin\theta\times OA$

$= W\sin\theta\times l$

$= W.\,l\sin\theta$

$= Wh,$

where h is the height of A above O.

∴ The total work done *against gravity*,

= work done against W cos θ *plus* work done against W sin θ

$= 0 + Wh$

$= Wh$

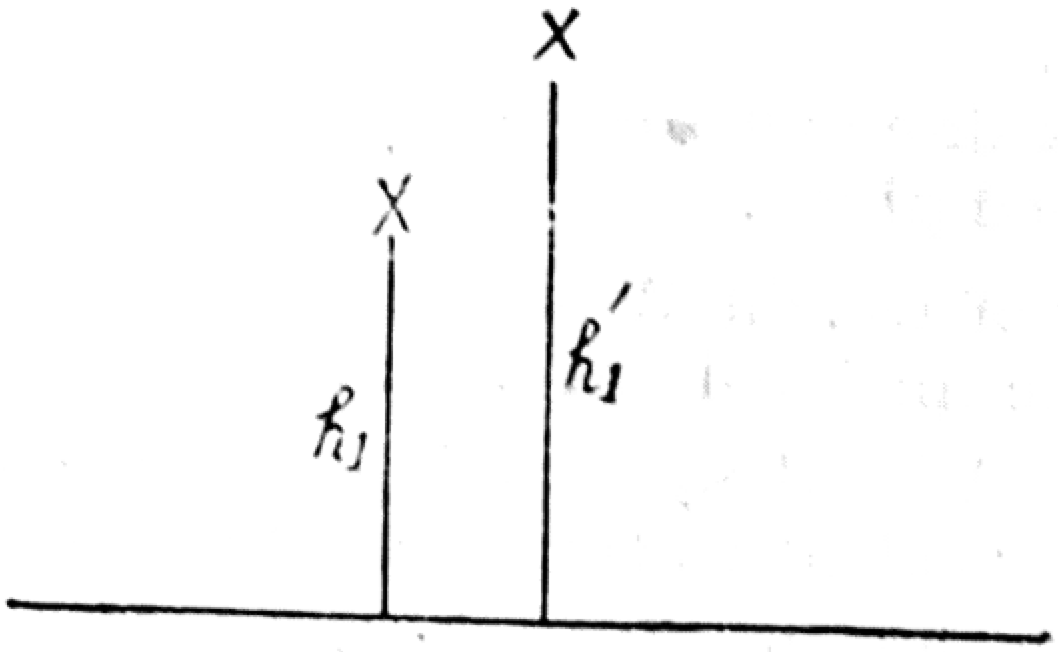
=(Weight of body) × (vertical distance through which the body has been raised).

Note.—If the body *slides down* the inclined plane, the work done by gravity is also equal in amount to $Wb\sin\theta$ or Wh.

Cor. If a body of weight W_1 be *raised* from height h_1 to height h_1' above the surface of the earth *by any path whatever*, then the *work done against gravity* is $W_1(h'^{-}{}_1h_1)$. For, this path may be supposed to be composed of a number of inclined planes for each of which the above formula would hold good. Hence *the total work done against gravity*

$= W_1(h'_1-h_1)$

$= W_1$ × (*total height through which the body has been raised*).



The following **theorem** follows at once from the corollary :

Theorem. *A system of particles is raised from one position to another ; to show that the work done against gravity is equal to Wh, where W is the total weight of the particles and h is the vertical distance between the final and the initial heights of the centre of gravity of the particles, i. e.,* work done $= W \times$ (final height of C. G *minus* initial height of C. G.)

Let W_1, W_2, W_3...... be the weights of the particles and
$h_1, h_2, h_3,$...... be their *intial heights* ; } above any
h'_1, h'_2, h'_3........... be their *final heights*, } horizontal plane.

Also let H and H′ be the *initial* and the *final* heights of the C. G. of the particles ; then

$$H=\frac{W_1h_1+W_2h_2+W_3h_3+\ldots}{W_1+W_2+W_3+\ldots(=W)},$$

i.e., $WH = W_1h_1 + W_2h_2 + W_3h_3 + \ldots$

and $$H'=\frac{W_1h_1'+W_2h_2'+W_3h_3'+\ldots}{W_1+W_2+W_3+\ldots(=W)},$$

i. e., $WH' = W_1h_1' + W_2h_2' + W_3h_3' + \ldots$

Now, the work done *against* gravity in raising the *first particle* is, by the corollary, equal to $W_1\ (h_1'-h_1)$; similarly, the work done in raising the *second, third*... particles are $W_2(h_2'-h_2)$, $W_3(h_3'-h_3)$...

Hence **the total work done against gravity** in raising the particles

$$=W_1(h_1'-h_1)+W_2(h_2'-h_2)+W_3(h_3'-h_3)+\ldots\ldots$$
$$=(W_1h_1'+W_2h_2'+W_3h_3'+\ldots\ldots\ldots)-(W_1h_1+W_2h_2+W_3h_3+\ldots)$$
$$=WH'-WH.$$
$$=W(H'-H).$$

$=W\times$ (*final height of C. G.* **minus** *initial height of C. G.*)

117. Rate of performing work : Power. The calculation of the amount of work done by an agent does not involve the question of time. But it is oftentimes required to do a certain amount of work *in a given time*. The measurement of *rate of work*, therefore, becomes necessary for many practical purposes. *The rate of doing work is called* **power.**

When the agent works uniformly,

$$\textit{Power} = \frac{\textit{Total amount of work}}{\textit{Time of doing it}}.$$

An agent is said to be working with *unit power if, unit work is done per second.* The unit of power is therefore either *1 foot-pound per second* or *1 foot-poundal per second.* These units are too small in many cases and larger units are employed. The larger unit used by Engineers is the **Horse Power**. An agent is said to be working with one *horse power* if it *performs 33000 foot-pounds of work per minute, or 550 foot-pounds per second.*

Ex. 1. *A body weighing 10 lbs. is thrown upward against gravity. Compute the work done upon it by its weight (a) while it rises 10 feet, (b) while it falls 10 ft.*

Solution. (*a*) While the body rises, it moves in a direction opposite to the direction of the force (its weight); therefore the work done by the weight is *negative*, and its amount = force × distance

$= 10 \times 10$ foot-pounds.

$\therefore$ The work done *by* the weight

$= -100$ foot-pounds.

(*b*) When the body falls 10 ft. the work done is positive and is equal to $+100$ foot-pounds.

Ex. 2. *The diameter of a vertical shaft which is full of water is 9 feet and its depth is 420 feet. Find the amount of work done in pumping the water to the level of the top of the shaft.*

Find the H. P. of the Engine which would empty the shaft in 9 hours and 48 minutes.

The water pumped out of the shaft will spread itself in the form of a thin sheet on the surface of the earth which is at the same level as the top of the shaft; the C. G. of this water will, therefore, be some where at this level; let it be G′.

(*i*) The final height of C. G. of water *minus* initial height of C.G. of the same

= height of G′ above G = 210 feet.

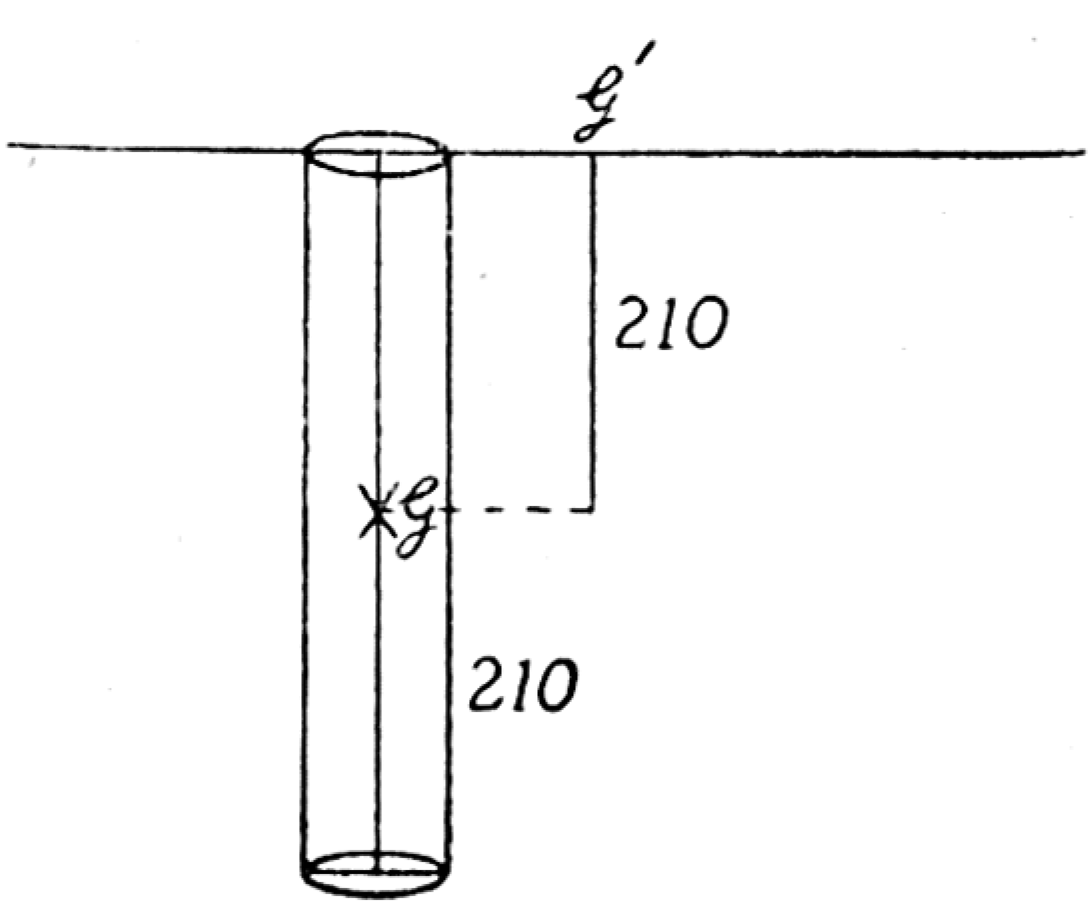
(*ii*) The *volume* of water $= \pi \times (\frac{9}{2})^2 \times 420$ *cubic feet.*

$\therefore$ its *weight* $=\pi\times(\frac{9}{2})^2\times 420\times 62\frac{1}{2}$ *pounds.*

(since it is known that a cubic foot of water weighs $62\frac{1}{2}$lbs.).

(*iii*) Hence, the total amount of work done by the engine

=Total weight of water lifted × height through which the C. G. of water has been raised



$=\pi\times(\frac{9}{2})^2\times 420\times 62\frac{1}{2}\times 210$ ft. pounds.

(*iv*) Again, 550 ft. pounds of work done *per second* counts for *one* H. P.

$\therefore$ If x be the H. P. of the engine, the work done by the engine in 9 hours 48 minutes (588×60 seconds) *is* $x\times 550\times 588\times 60$ foot pounds of work and this must be equal to $\pi\times(\frac{9}{2})^2\times 420\times 62\frac{1}{2}\times 210$ ft. lbs.

$$\therefore\quad x\times 550\times 588\times 60=\pi\times\frac{81}{4}\times 420\times\frac{125}{2}\times 210.$$

$$\therefore\quad x=\frac{22\times 81\times 420\times 125\times 210}{7\times 4\times 2\times 550\times 588\times 60}\qquad =18\frac{9}{112}\text{ H.P.}$$

Exercises XIV

1. A man is cycling on level ground at the rate of 15 miles per hour ; if the resistance to his motion be 6 lbs. wt., find the work done in an hour, and the H. P. at which he is working.

2. A man weighing 12 stones clmibs a hill so that he goes up through a height of 660 ft. in one hour. Find the hours power at which he is working.

Show that if a number of men go up at the same rate, the horse-power at which each works would be proportional to his mass.

3. A cyclist goes up a hill road of uniform gradient, his weight including that of the machine being 15 stones and the resistance of the road being 5 lbs. wt. If the road has a rise of 1 in 50, find the work done in going a distance of one mile.

4. Find the work done in digging a well 77 ft. deep when the uniform diameter of the horizontal section is 8 ft. the soil whose average weight is 160 lbs. per cu. ft. being spread on the surface of the ground above,

given that $\left(\pi = \frac{22}{7}\right)$.

5. Find the work done in emptying a tank full of water, whose rectangular horizontal section is 50 ft. by 21 ft., and depth 12 ft., the top of the tank being at the level of the ground.

Find the time taken by an engine of one H. P. to do the work.

6. Find the H. P. of an engine that would empty in 25 hours a cylindrical shaft full of water whose diameter is 7 ft. and depth 240 ft.

7. Find the resistances to the motion of a motor-car with an engine of 25 H. P. if it can develop a maximum speed of 40 m.p.h. on a level road.

8. A motor-car can attain a maximum speed of 25 m.p.h. on a hill-road whose slope is 1 in 20. Find the least horse power of its engines if the total load including the weight of the car be $\frac{3}{4}$ of a ton, the resistances of the road etc. being equivalent to 180 lbs. wt.

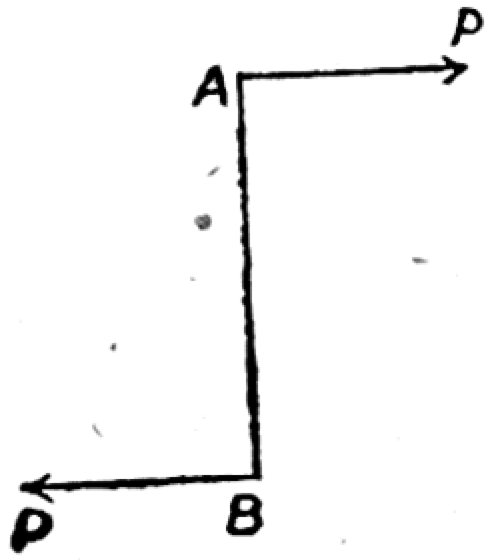
9. Find the least horse power of an engine which can just pull a train of m tons with velocity v, m.p.h. up an incline of 1 in n, the resistance to motion being x lbs. per ton on level.

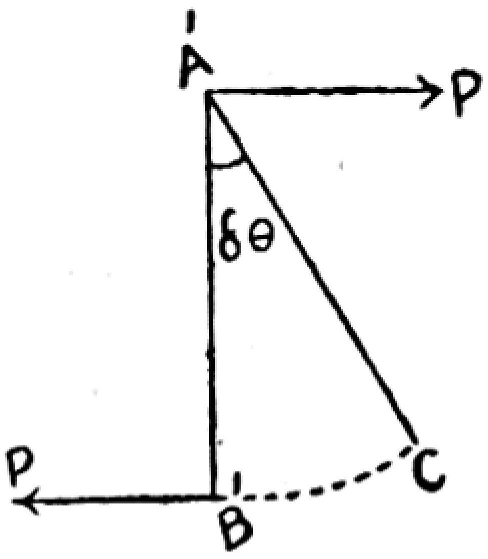
If $v=40$ m.p.h. when $x=15$, $n=500$, show that the engine will be able to pull the train down the same incline at 74 m.p.h. approximately. *(P.U. 1924)*

10. A car is travelling at its maximum speed of 40 miles per hour on the level, the resistance being 160 lbs. wt. per ton, assumed to be independent of the speed. It then climbs a hill, gradient 1 in 25, and the speed falls until it is steady, the engine then working at the same effective H. P. as before. Find its steady speed up the hill. *(P. U. 1926)*

117. A. Work done by a couple.

Let the couple (P,P) act at an arm AB. If M denote the moment of the couple M = P.AB.





[CB'=A'C×δθ]

Suppose arm AB to move to the position A'B' and let the small angle between the arms in these two positions be $\delta\theta$.

Arm AB may be supposed to take up the posision A'B' in two stages.

(*i*) AB moves parallel to itself and takes up the position A'C. The forces of the couple move parallel to themselves.

(*ii*) A'C rotates through $\delta\theta$ about A' and takes up the position A'B'.

In the first operation, the work done by equal and opposite forces P and P is *zero.*

In the second operation, the force P at A' *does no work* since its point of application does not move.

But P at C displaces the point C from C to B' the displacement being A'C, $\delta\theta$ and the work done is $P \times A'C.\ \delta\theta$.

$\therefore$ *Total work done in these* two operations$=P.\ A'C.\delta\theta$

$=P.AB.\delta\theta=M_1\delta\theta=$ *Moment of the couple multiplied by the small angle of rotation.*

Note. The work done in turning through the finite

$$\text{angle } \theta=\int_0^\theta M\, d\theta$$

If M remains constant, then

$$\text{work done}=M\int_0^\theta d\theta=M\theta$$

$=$ *Moment of couple multiplied by the angle turned through.*

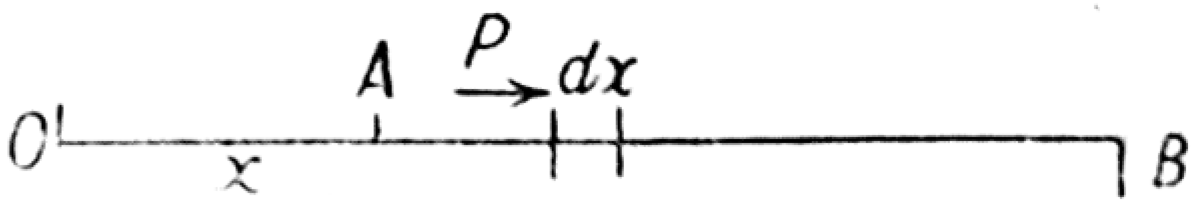
VARIABLE FORCE

118. Work done in a finite displacement by a variable force.

If a force be *variable*, we cannot obtain the work over the *whole path* by applying the formula *work = force × displacement*. In such cases, *we divide the whole path* (*straight or curved*) *into an infinite number of small parts*. We can then apply the formula to each of these small portions of the path, since average force over any of these small paths may be considered to be constant. We shall consider the cases when the displacement takes place along a straight or curved path.

Case (i) *Variable force acting along the (straight) line of displacement.*

Let a body or a particle move along the straight line OX, the force at distance x from the origin O being P which is a function of x.



Let the particle move from distance x to distance $x+dx$, supposing the force P to remain constant over this small element of path. The small amount of work dW done by P is approximately equal to Pdx, so that

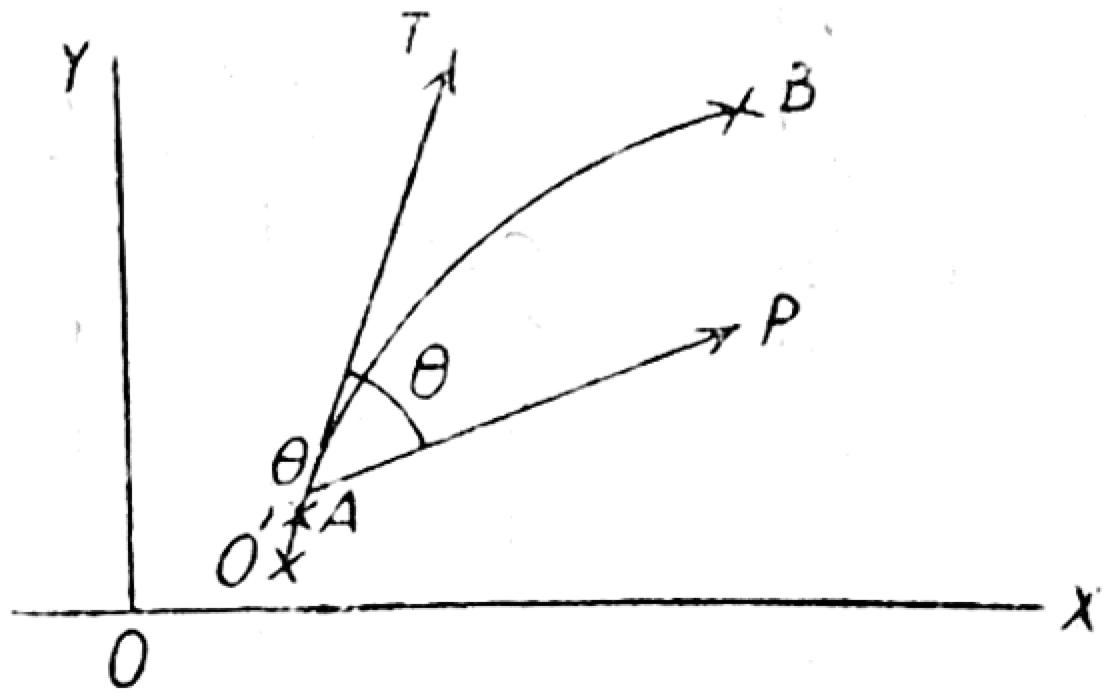
$$dW = Pdx.$$

∴ The total work done is the sum of such small *elements of work* Pdx.

Hence the total work done by P when the displacement is from $x=a$ to $x=b$, is, by the *definition of definite integral as summation*, given by

$$W = \int_a^b Pdx.$$

Case (ii) *Variable force acting on a body moving in a curve.*



Let a body move in a plane along the path O'AB, (O' being the point from which the arc is measured) under the action of a force P which is variable both in magnitude and direction. Divide the path O'AB into a larg number of indefinitely small elements, and let *ds* be the length of one of these small elements of the path. Then the *element of work dw* is approximately equal to P multiplied by *ds* cos θ, θ being the angle between the direction of the force and the tangent to the curve. Then

$$dW = P \times ds \cos\theta.$$

∴ the total work done from s_0 to s is given by

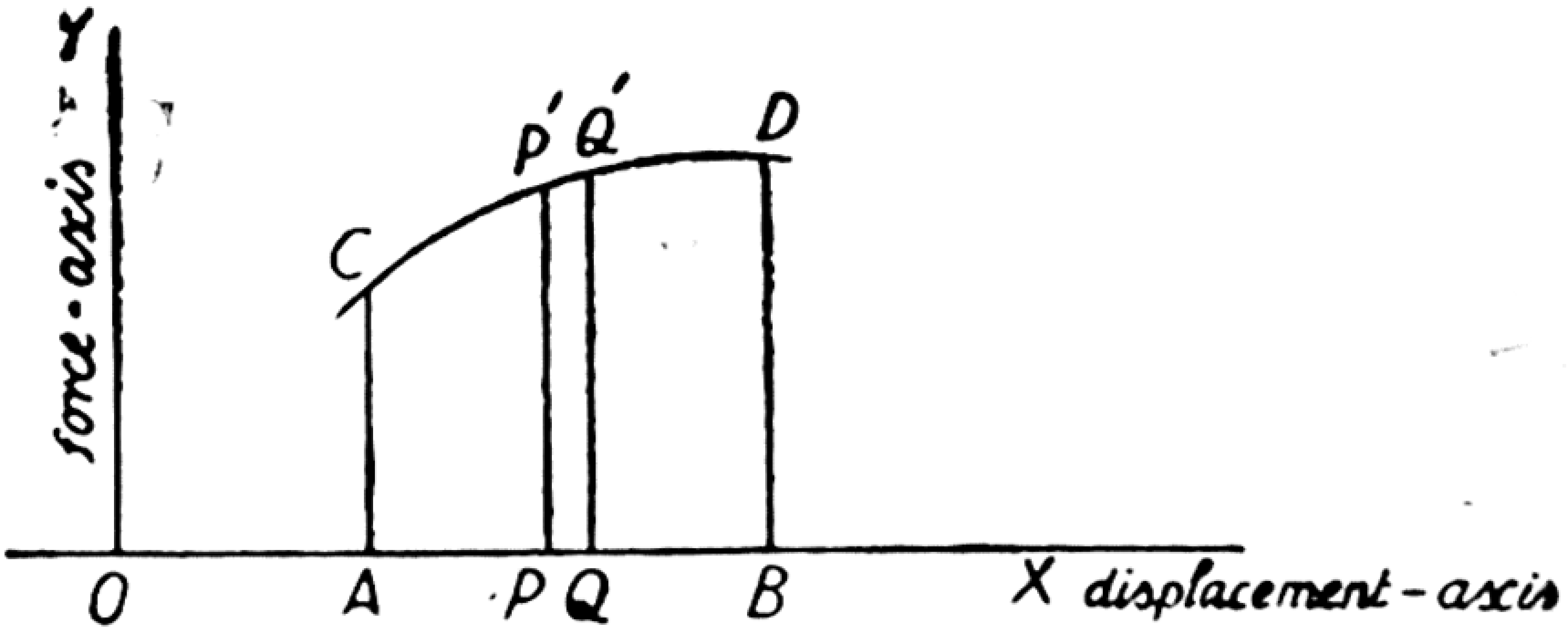
$$W = \int_{s_0}^{s} P \cos\theta \, ds.$$

119. Graphical method for finding the work done by a variable force acting along a straight line.

Work represented by an area. We have already seen in Art. 118, Case (*i*), that the work done by a *variable force* acting along a straight line is represented by the definite integral $\int_a^b P dx$. For the convenience of beginners unacquainted with Integral Calculus, the following *geometrical method* of finding the work done is useful.

Let AB represent the path described by a moving body under the action of a force actine along OX ; let ordinates be drawn at each point in AB to represent the force at that point. The extremities of the ordinates will form the curve CP'Q'D, PP' and QQ' being two neighbouring ordinates at the adjacent points P and Q.

We shall *prove that the work done by the variable force as the point of application moves from A to B is represented by the arc ACDBA.*



Now, the small area $PP'Q'QP = PP' \times PQ$, in the limit
$=$ displacement $PQ \times$ force
$=$ work done in moving the body from P to Q.

$\therefore$ area $ACDBA =$ *sum* of such small areas $PP'Q'QP$
$=$ work done in moving the body from A to B.

CHAPTER X

*VIRTUAL WORK

120. In the chapter on Work, the displacement produced in a body under the action of forces is considered to be that which the body *actually* undergoes. In Statics, *if a body is in equilibrium*, there is *no motion*, and consequently there is no *actual displacement.* But it is found convenient in some statical investigations to allow the body to receive an *imaginary displacement* called **virtual displacement**. Whether the displacement is *actual* or *virtual*, (imaginary), each of these forces will do work in accordance with the technical definition of " work " even where the forces acting on the body are in equilibrium. The work done by a force in such an imaginary displacement is called *virtual work.*

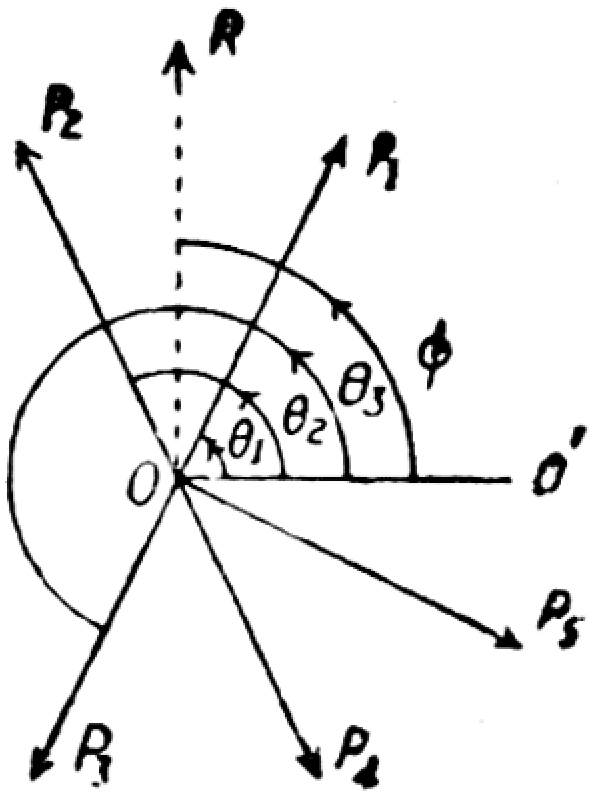
Definition. *The* **virtual work** *of a force is the product of the force and the projection along its direction of the virtual displacement of its point of application.*

The conception of virtual work enables us to formulate an important principle of great *usefulness* in Statics called the *principle of virtual work.*

The teacher, if he so chooses, can take up this chapter at a later stage.

121. *If a particle acted on by any system of coplanar forces receive any virtual displacement whatever, the algebraic sum of the virtual work done by the different forces during the displacement is equal to the virtual work done by their resultant.*

Let the point of application of the forces be displaced from O to O′, let forces P_1, P_2, P_3, ...(*constant in magnitude and direction, but not in line of action,*) act on the particle in directions which make angles θ_1, θ_2, θ_3,... with OO′. These forces, since they act at a point, must have a resultant. Let R be the resultant and let it make an angle ϕ with OO′. The virtual work done by $P_1 = P_1 \times OO' \cos\theta_1$; similarly, the virtual work done by P_2, P_3, ...are $P_2 \times OO' \cos\theta_2$, $P_3 \times OO' \cos\theta_3$... Hence the algebraice sum of the virtual work done by the component forces P_1, P_2, $\ldots\ldots = P_1 \times OO' \cos\theta_1 + P_2 \times OO' \cos\theta + P_3 \times OO' \cos\theta_3 + \ldots\ldots\ldots\ldots\ldots = OO' \times {}_2 (P_1 \cos\theta_1 + P_2 \cos\theta_2 + P_3 \cos\theta_3 + \ldots) =$ $OO' \times$ algebraic sum of the resolved parts of the forces in the direction $OO' = OO' \times$ resolved part of the resultant R in the direction OO′



$$= OO' \times R \cos\phi$$

$$= R \times OO' \cos\phi$$

= *Virtual work done by the resultant.*

This proposition is true whether the displacement is virtual or actual.

Note 1. Observe that the displacement OO′ in the case of a particle is *perfectly arbitrary both in magnitude as well as in direction.* The proof of the proposition would have been exactly similar if we had chosen any other direction or length for the displacement OO′.

Note 2. In order to secure that the forces may not sensibly alter during the virtual displacement, it is usually taken, *very small.*

Note 3. If the particle be in **equilibrium,** then $R = 0$: the work done by the resultants, therefore, zero, and we deduce the following **corollary :**

If a particle be in **equilibrium** under the action of any number of co-planar forces and if the particle receive any virtual displacement, the algebraic sum of the virtual work done by different forces is zero. This is **the principle of virtual work for a particle.**

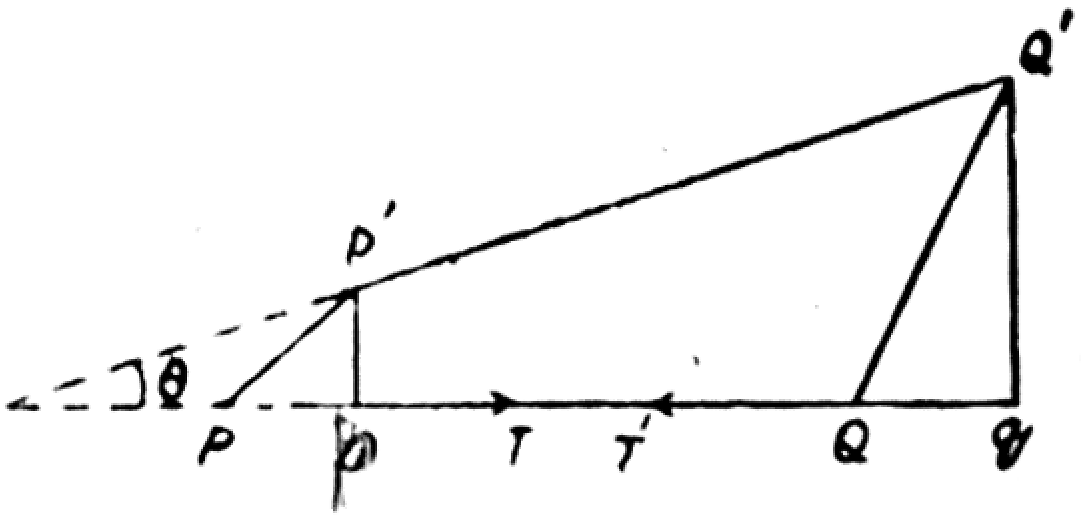
The converse of this principle is also true, for when the equation of virtual work is formed for displacement, in two different directions we obtain $\Sigma P\,co\alpha=0$, $\Sigma P \cos\beta=0$, whence it follows that $R=0$.

122. Principle of Virtual Work for a System of Particles. If a system of particles is *in equilibrium* under the action of *external and interal forces*, and if the system receive any *virtual displacement* whatever, the virtual work done by the forces on each particle is nil (*Note* 2 *Art.* *121*). Therefore, **the algebraic sum of the virtual work done by all the forces, external and internal,** *is zero.*

123. Elimination of the internal forces of the system from the equation of virtual work.

Theorem. *If a system of* **two particles** *rigidly connected together be* **in equilibrium** *and if the system receive an indefinitely small virtual displacement, the virtual work done by the action and reaction between the two particles is zero.*

Let P, Q be the original and P′, Q′ the new positions of the particles after the displacement, and let P′Q′ be inclined to PQ at an angle θ which, by hypothesis, is indefinitely small. Let the pair of forces, which constitute action and reaction between the two particles P and Q and which are equal and opposite by Newton's third law, be denoted by T and T′. We shall consider them to be *pulls* so that P is pulled towards Q in the direction PQ by a force T and Q is pulled towards P in the direction QP by a force T′. Draw P′p and Q′q perpendiculars from P′ and Q′ on PQ. Now,



Virtual work done by $T=T\times Pp$.

Virtual work done by $T'=-T'\times Qq$.

$\therefore$ *Total virtual work done by action and reaction*

$$=T\times Pp-T'\times Qq$$
$$=T(Pp-Qq) \quad [\text{since } T=T',$$
$$=T\{(Pp+pQ)-(Qq+pQ)\}$$

[adding and subtracting pQ,

$$=T(PQ-pq)$$

$=T(PQ - \text{projection of } P'Q' \text{ on } PQ)$ ......(A)

But projection of $P'Q'$ on $PQ = P'Q' \cos\theta$

$=PQ\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \cdots\right)$ $\because$ $P'Q' = PQ$, since particles are rigidly connected.

$=PQ$, *neglecting* θ^2, θ^4..............................etc., *since* θ *is small.*

Hence *total virtual work done by action and reaction*
$=T(PQ - PQ)$ from (A)
$=0$.

Note 1.—If the action and reaction be *pushes* instead of *pulls* we shall have the same result.

Note 2.—**Work done by the tension of an inextensible string.** We must imagine the string to be stretched in spite of its actual inextensibility so that work may be done by the string. In this case we cannot take $P'Q' = PQ$. Hence work done by the tension is by (A) $= T(PQ - \text{projection of } P'Q')$

$=T(PQ - P'Q' \cos\theta)$
$=T(PQ - P'Q')$ since θ is small
$=-T(P'Q' - PQ)$
$=-T \times$ *increase in length of string.*

124. Theorem Extended to a system of particles. If, instead of two particles, we have a system consisting of any number of particles rigidly connected together and *if this system be in equilibrium, the* **algebraic sum of the virtual works done by all the internal forces is zero.** *For*, the internal forces, arising from the internal connections of the parts of the system among themselves, are directed from particle to particle and consist entirely of pairs (each pair being made up of an action and the corresponding reaction) and we have just seen that *the virtual work done by each pair is zero* (Art. 123).

125. Now, the algebraic sum of the virtual work done by *all the forces, external and internal*, is zero (Art. 122). Also the algebraic sum of the virtual work done by the internal forces is zero (Art. 124). Therefore **the algebraic sum of the virtual work done by the external forces alone is zero.** We thus see that *in forming the equation of virtual work for a system of particles in equilibrium, the internal forces need not be taken into consideration.* We thus arrive at the important, *principle* which may be stated as follows.

" *If a system of particles* **in equilibrium** *under the action of any system of external forces together with internal forces, receive any* **indefinitely small virtual displacement** *whatever, which does not alter the configuration formed by the particles,* (which does not alter the shape and size of the figure formed by the particles *i.e.*, the system may be considered as rigidly connected), *the algebraic sum of the virtual works done by the external forces alone is zero.* (Or more strictly speaking, is of *an order of smallness higher than that of the displacement. See Art. 123*).

This is the Principle of virtual work for a system of particles.

The converse of this principle may also be taken as true.

Remarks. We have been enabled to *exclude the internal forces* from the equation of virtual work by choosing a *particular type of virtual displacement for the system of particles in equilibrium.* (i) The virtual displacement chosen must be an **indefinitely small displacement**, (so that θ^2, θ^4 may be neglected; Art. 102). (ii) In the second place, the virtual displacement chosen must be compatible *with the geometrical conditions of the system* so that the **mutual distances of the particles remain unaltered.**

126. Principle of virtual work for a rigid body and a system of bodies.

(*i*) It is evident from what has been said above that we can apply the *principle of virtual work* to a rigid body if we suppose (i) the displacement to be indefinitely small and if we suppose that (ii) the displacement is such that the body moves as a whole *without change of shape or size.*

(*ii*) *When any number of rigid bodies are joined together by inextensible strings or rods*, there is an action between one part of the system and another and an equal and opposite reaction of the second part on the first and by Art. 123, it is easily seen that the virtual work contributed by this pair of forces is zero.

(*b*) Again, if *one part of the system move over a smooth surface*, the work performed by the pressure is zero as the point of contact is moving tangentially to the direction of the pressure which acts in the normal direction.

(*c*) Similarly, if *two bodies of the system are smoothly hinged together at one point* and if the displacement be so chosen that the joints remain unbroken, the work done by the action and reaction at the joint is zero. These considerations enable us to apply practically the *principle of virtual work* to the solution of problems dealing with the equilibrium of a system of rigid bodies connected together. Any detailed investigation of the work done in the displacement of a system of bodies by the *two types of forces, internal and external,* is beyond the scope of the present work. It will be enough to remember that in calculating the work done in a small displacement *a great number of terms arising from the work of the internal forces can be left out of account.*

The principle of virtual work for any rigid body is sometimes stated in the following manner.

" *If a system of forces acting on a body be in equilibrium and the body undergo a slight displacement consistent with the geometrical conditions of the system, the algebraic sum of the virtual works is zero*, and **conversely,** *if this algebraic sum be zero, the forces are in equilibrium.*" The converse theorem whose truth is taken for granted is implied in the principle of virtual work.

Summary. We thus notice that in the application of the principle of virtual work to a connected system of rigid bodies, certain reactions between the bodies, and certain external constraints may be ignored.

For virtual displacements which do not violate the geometrical conditions of the system, the following forces may be omitted as they contribute nothing to the equation of the virtual work of the system.

1. *The tensions of inextensible strings or of rods, connecting the bodies of the system.* (Art. 123).

2. *The reaction of any smooth surface with which the bodies are in contact.*

3. *The reaction at any point of contact with a fixed surface on which the body rolls without sliding.* For, the point of contact is instantaneously at rest, hence the normal reaction and the friction at this point have zero displacement and the work done by each is nil.

4. The reactions between any two bodies of the system under consideration.

This follows from the fact that action and reaction are equal and opposite and therefore, the virtual works contributed by them cancel each other.

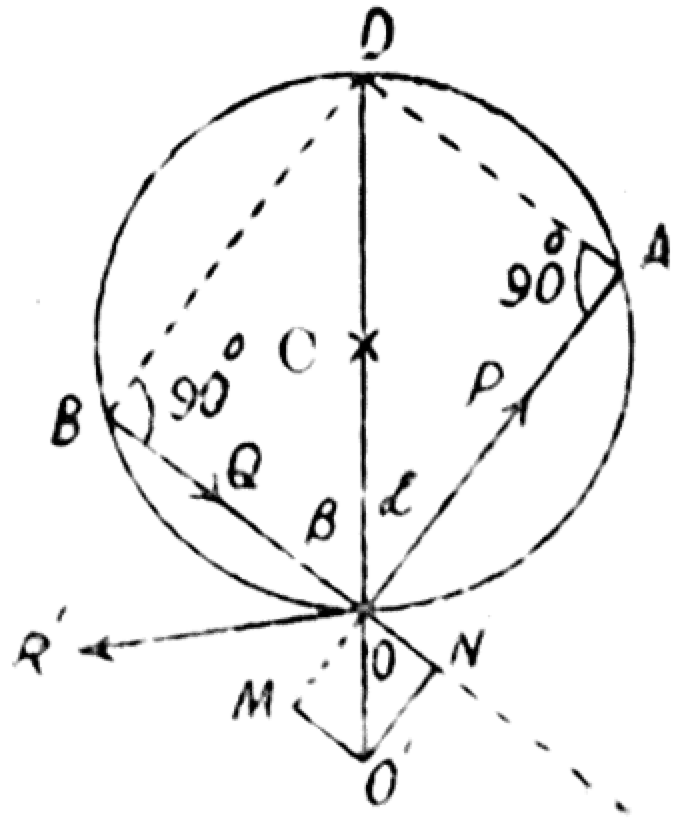
Exception :—If the point of contact of the bodies be also in motion and the bodies be rough, then the work contributed by the action and reaction will not cancel each other.

Note 1. "The connections of the various parts in a system with one another, and the restraints imposed by strings or contact with smooth planes etc., are usually spoken of as the geometrical conditions of the system."

Note 2. In solving problems on the equilibrium of bodies, it is sometimes convenient to choose a *displacement which violates some of the geometrical conditions. This is allowable if we include the work of the internal forces in the equation of virtual work.* (See Note 2 Art. 123).

Worked Examples

127. Ex. 1. *O is a point on a circle and two forces inversely proportional to any two chords, OA and BO act at O in the senses OA and BO ; prove that their resultant acts along the tangent to the circle at O.*



Solution. Let forces P and Q act at O along OA, BO inversely proportional to these chords so that

$$P = \frac{K}{OA} \text{ and } Q = \frac{K}{BO}.$$

Let $\angle COA = \alpha$ and $\angle COB = \beta$, where C is the centre of the circle.

The forces P and Q *must have a resultant* ; there is no question of equilibrium under two forces unless these forces are equal and opposite. Introduce a force R′ at O equal to the *opposite of the resultant* of P and Q. Then the *particle at O is in equilibrium* under the action of the three forces P, Q and R′.

Let O receive a virtual displacement along CO, so that O′ is the new position of O. By the *principle of virtual work, the algebraic sum of the virtual work done by* P,Q and $R'=0$. (A) Now, the *virtual work done by* $P=-P\times OM=-P\times OO'\cos\alpha$ and the virtual work done by $Q=+Q.ON=Q\times OO'\cos\beta$.

(B) ∴ *the virtual work done by* P and Q

$$=-P\times OO'\cos\alpha+Q\times OO'\cos\beta$$

$$=OO'\left\{\frac{-K}{OA}\cos\alpha+\frac{K}{BO}\times\cos\beta\right\}$$

$$=\frac{OO'\times K}{OA\times BO}\left\{-BO.\cos\alpha+OA\cos\beta\right\}$$

$$=\frac{OO'\times K}{OA\times BO}\left\{-(DO\cos\beta)\cos\alpha+(DO\cos\alpha)\cos\beta\right\}$$

$$=0.$$

∴ from (A) and (B),

the virtual work done by $R'=0$

i.e., R' is ⊥ OO′ (the direction of the diameter)

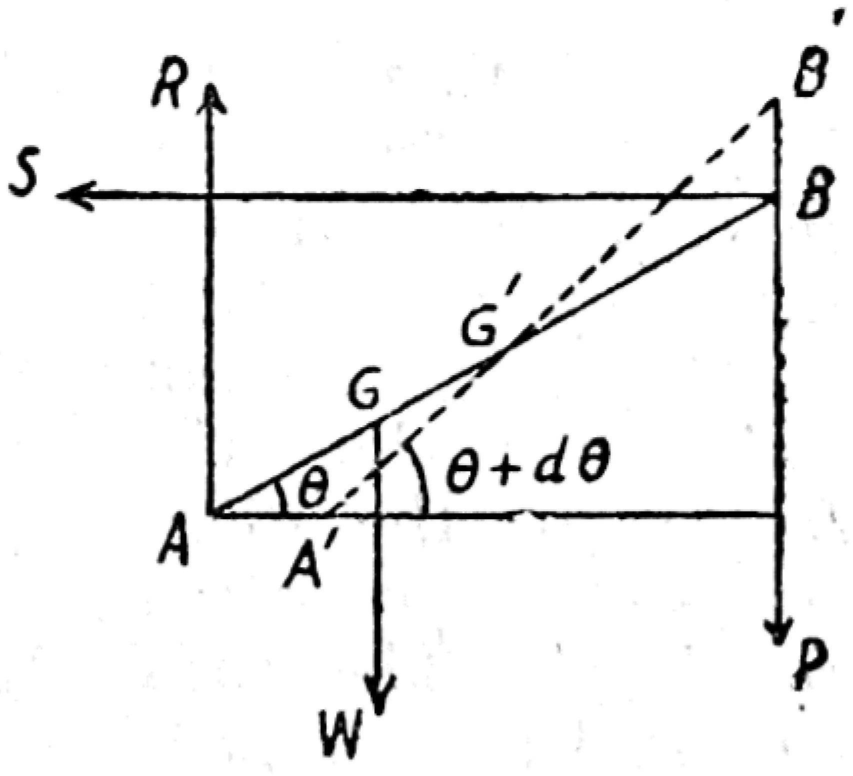
∴ R′ acts along *tangent*

i.e., the opposite of the resultant acts along the tangent.

Hence, the resultant of P and Q acts along the tangent to the circle at O.

Ex. 2. *A heavy beam AB rests against a smooth horizontal plane, CA, and a smooth vertical wall, CB, the lower extremity A being attached to a cord which passes over a smooth pulley at C and sustains a given weight P. Find the position of equilibrium and the pressures on the plane and wall.*

Solution. Let θ be the inclination of the beam to the horizon in the position of equilibrium; let G, the C. G. of the beam be at a distance a from A and b from B. Let the weight of the beam be W. The reactions R and S at A and B are normal to the plane and the wall since these surfaces are smooth. Let the beam receive an imaginary displacement so that the ends A and B remain in contact



with the planes. Let the displaced position A′B′ of the beam make an angle $\theta+d\theta$ with the horizontal plane. Then *by the principle of virtual work, the algebraic sum of the virtual work done by the forces, P, W, R, and S is equal to zero.*

But the virtual works done by R and S are zero, since the displacements AA′ and BB′ are perpendicular to R and S.

Hence, *algebraic sum of work done by P and W=0.* Now, *work done* by $P = P \times AA'$

$$= P \times (a+b)\ \{ \cos\theta - \cos(\theta+d\theta) \}$$
$$= P \times (a+b) \sin\theta.\ d\theta.$$

(Since $\cos d\theta = 1$ and $\sin d\theta = d\theta$)

Work done by W is negative and is

$$= W \times \text{distance through which G has been raised}$$
$$= W.\ a \{ \sin(\theta+d\theta) - \sin\theta \}$$
$$= W.\ a \cos\theta.\ d\theta.$$

$\therefore$ work done by $W = -Wa\cos\theta.\ d\theta.$

Hence,

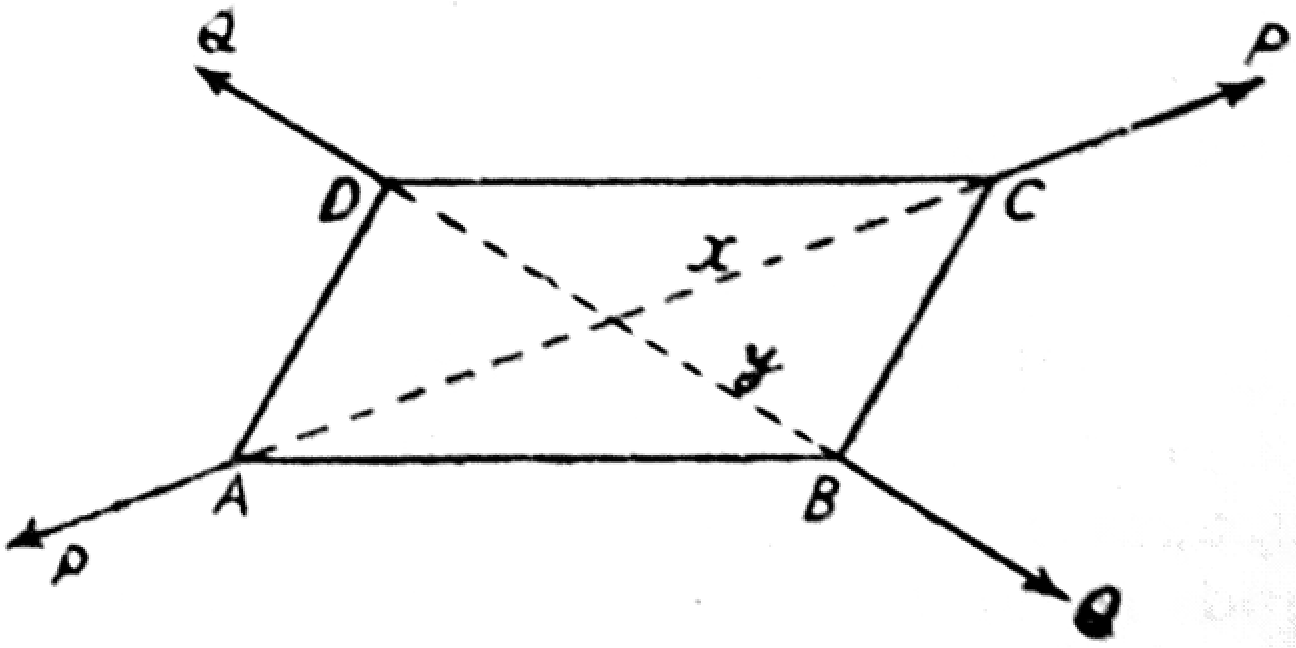
$$-Wa\cos\theta\ d\theta + P(a+b)\sin\theta.\ d\theta = 0$$

i.e. $Wa\cos\theta = P(a+b)\sin\theta,$

$$\therefore \tan\theta = \frac{Wa}{P(a+b)}.$$

Ex. 3. *Four fine light rods are jointed together at their extremities to form the parallelogram ABCD. Equal and opposite forces P act along one diagonal AC and equal and opposite forces Q act along the other diagonal BD, all the forces being directed outwards. Prove that in order that the frame may be in equilibrium, P and Q are proportionate to the length of the diagonals along which they act.*

Solution. Let x and y be the lengths of the diagonals. Let the frame be given a slight virtual displacement so that the joints remain unbroken; let the lengths of the diagonals after distortion be x', y'. The rods will still form a par-



allelogram. As the sum of the squares on the sides of a parallelogram is equal to the sum of the squares on its diagonals, we have

$$x'^2+y'^2=x^2+y^2$$

$$\therefore\ x'^2-x^2=-(y'^2-y^2)$$

$$\therefore\ (x'-x)(x'+x)=-(y'-y)(y'+y) \qquad \ldots(1).$$

Since the joints are smooth, no work is done by the action and the reaction at each of these joints. Hence, *by the principle of virtual work the algebraic sum of the work done by the 4 forces*$=0$.

But, the work done by equal and opposite force P and $P=P\,(x'-x)$ (as in Note 2, Art. 123). and work done by equal and opposite forces Q and $Q=Q(y'-y)$

$$\therefore\ P(x'-x)+Q(y'-y)=0 \qquad \ldots(2)$$

By transposing and dividing the two equations (1) and (2), we get

$$\frac{P}{Q}=-\frac{y'-y}{x'-x}=+\frac{x'+x}{y'+y}$$

But in the limit when x' approaches x and y' approaches y, we get

$$\frac{x'+x}{y'+y}=\frac{2x}{2y}=\frac{x}{y}$$

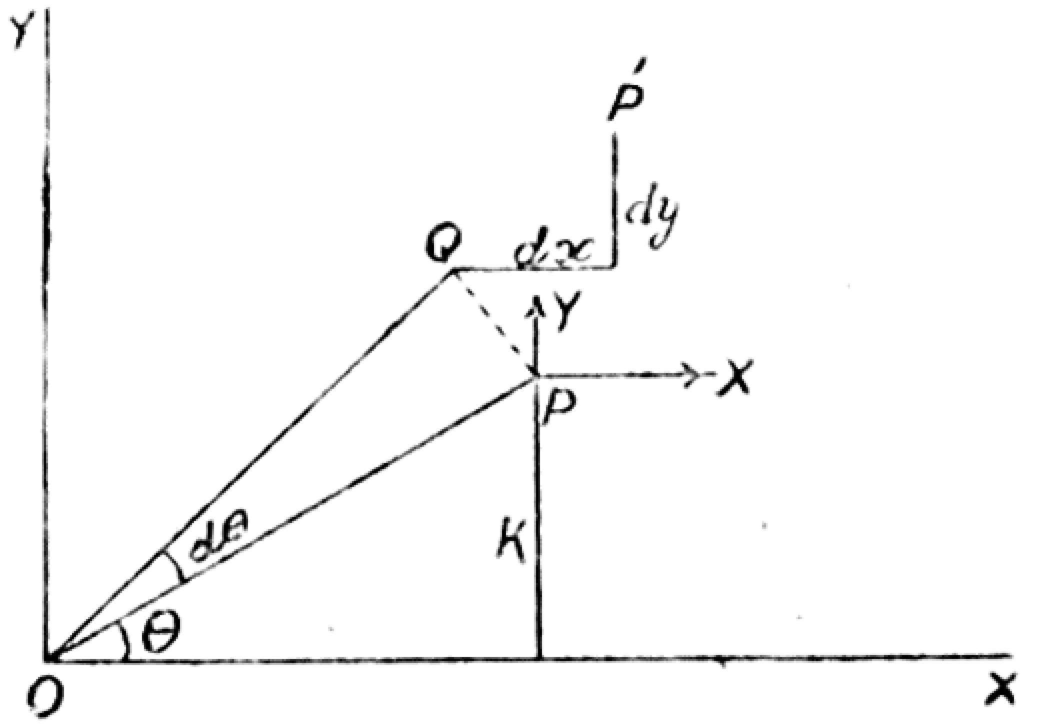
$$\therefore\ \frac{P}{Q}=\frac{x}{y}\ \textit{i.e.,}\ \frac{P}{x}=\frac{Q}{y}.$$

Exercises XIV

1. A force R having components X, Y along the rectangular axes x, y acts on a particle at P (h, k) and causes it to move to an infinitesimally near position P′. If dx, dy denote the linear displacements of P parallel to the axes and $d\theta$ its angular displacement, prove that work done$=X\,dx+Y\,dy+(Yh-Xk)\,d\theta$.

Deduce the equation of virtual work for a coplanar system of forces (*the sum of the virtual works of a system of coplanar forces which is in equilibrium is zero*) and establish its converse.

The particle P may be moved to P′ by first turning it through $d\theta$ about O and then moving it through dx and dy parallel to Ox and Oy.



Let (r, θ) be the polar co-ordinates of P, so that OP$=r$ and $\angle x$OP$=\theta$.

The Cartesian Co-ordinates of P′ are r cos $(\theta+d\theta)+dx$ and r sin $(\theta+d\theta)+dy$.

i.e., $r\cos\theta\cos d\theta$ $-r\sin\theta\sin d\theta+dx$ and $r\sin\theta.\cos d\theta+r\cos\theta.\sin d\theta+dy$.

(*i*) or $r\cos\theta-r\sin\theta\ d\theta+dx$ and $r\sin\theta+r\cos\theta\ d\theta+dy$.

The Cartesian Co-ordinates of P are

(*ii*) $r\cos\theta\ (=h)$ and $r\sin\theta\ (=k)$

∴ Their differences are

$-r\sin\theta\ d\theta+dx$ and $r\cos\theta\ d\theta+dy$.

(*iii*) or $-kd\theta+dx$ and $hd\theta+dy$.

Which are, therefore, the displacements of the particle in the directions of the forces X and Y.

Hence the work done$=\mathrm{X}(-\mathrm{K}d\theta+dx)+y\ (hd\theta+dy)$
$=\mathrm{X}dx+\mathrm{Y}dy+(\mathrm{Y}h-\mathrm{X}k)\ d\theta$.

It follows that if *a system of* **coplanar forces** acting on a *body* be in equilibrium and the body undergo *a slight displacement*, the *algebraic sum of the virtual works* is zero provided that

$$dx\,\Sigma\mathrm{X}+dy\,\Sigma\mathrm{Y}+d\theta\,\Sigma(\mathrm{Y}h-\Sigma k)=0 \qquad \text{......(1)}$$

Since the forces are in equilibrium, we have by Art. 57 $\Sigma\mathrm{X}=0$, $\Sigma\mathrm{Y}=0$ and $\Sigma(\mathrm{Y}h-\mathrm{X}k)=0$; the last show that the algebraic sum of the moments of the forces about the origin is zero.

Conversely, if the sum of the virtual works be zero for all displacements, the forces are in equilibrium.

Now (1) holds for *any displacement.* Taking $dy=0$, $d\theta=0$ we have $\Sigma\mathrm{X}=0$. Similarly $\Sigma\mathrm{Y}=0$, $\Sigma(\mathrm{Y}h-\mathrm{X}k)=0$.

∴ the forces are in equilibrium.

2. ABCD is a square frame work of equal uniform rods each of weight W, suspended from the point A and kept in shape by a light string AC. Calculate the tension of AC.

3. A smooth horizontal rail is parallel to a smooth wall and one foot from it, a uniform beam 16 ft. long rests over the rail with one end pressing against the wall. Find the angle between the beam and the wall in the position of equilibrium.

4. Four equal rods are jointed together to form a rhombus, and the opposite joints are joined by strings forming the diagonals and the whole system is placed on a smooth horizontal table. Show that their tensions are in the same ratio as their lengths.

5. Two equal uniform rods AB and AC each of length $2b$, are freely jointed at A and rest on a smooth vertical circle of radius a. Show that if 2θ be the angle between them

$$b \sin^3\theta = a \cos \theta.$$

6. A tripod is formed of three uniform rods, each of length $2a$ and weight W, freely jointed together at the vertex. It stands on a smooth horizontal plane, and is prevented from collapsing by equal strings of length $a\sqrt{3}$ joining the feet. Prove that, where a weight W is hung from the vertex, the tension in each string is $\frac{W}{6} + \frac{W}{9}$.

7. If in Ex. 5, the tripod is prevented from collapsing by strings of length $2b$ joining the middle points of the rods, prove that the tension in each string is

$$\tfrac{2}{3}(2W+3W)\,\frac{b}{\sqrt{9a^2-12b^2}}.$$

8. A regular hexagon ABCDEF is composed of six heavy rods, each of weight W, freely jointed together and suspended from the joint A, two light stiff storts CE and BF being inserted to prevent change of shape. Find the forces in CE and BF.

CHAPTER XI

FRICTION

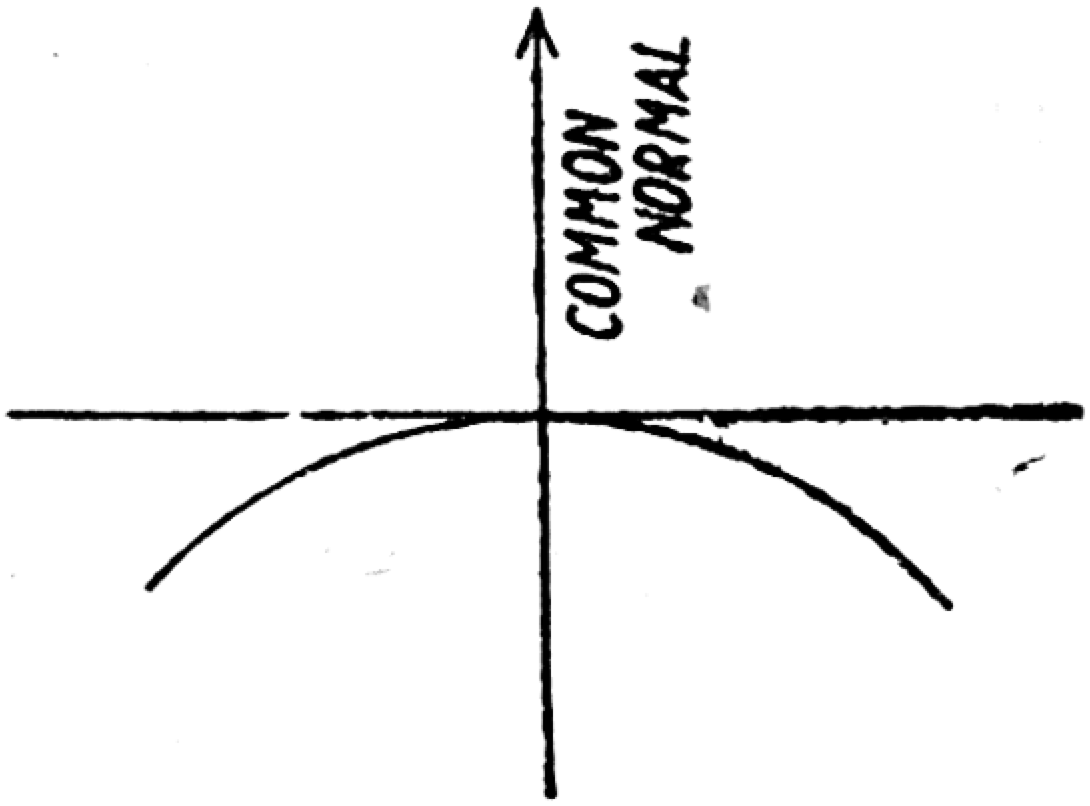
128. Rough and smooth bodies. If we attempt to drag a heavy body along the ground by means of a horizontal force, a resistance is felt to the motion of the body. This resistance (called *friction*) is a tangential force which opposes the applied horizontal force, the tangential resistance being generated on account of the roughness of the ground. If the ground were perfectly **smooth**, no such tangential resistance would have been felt (see art 13)

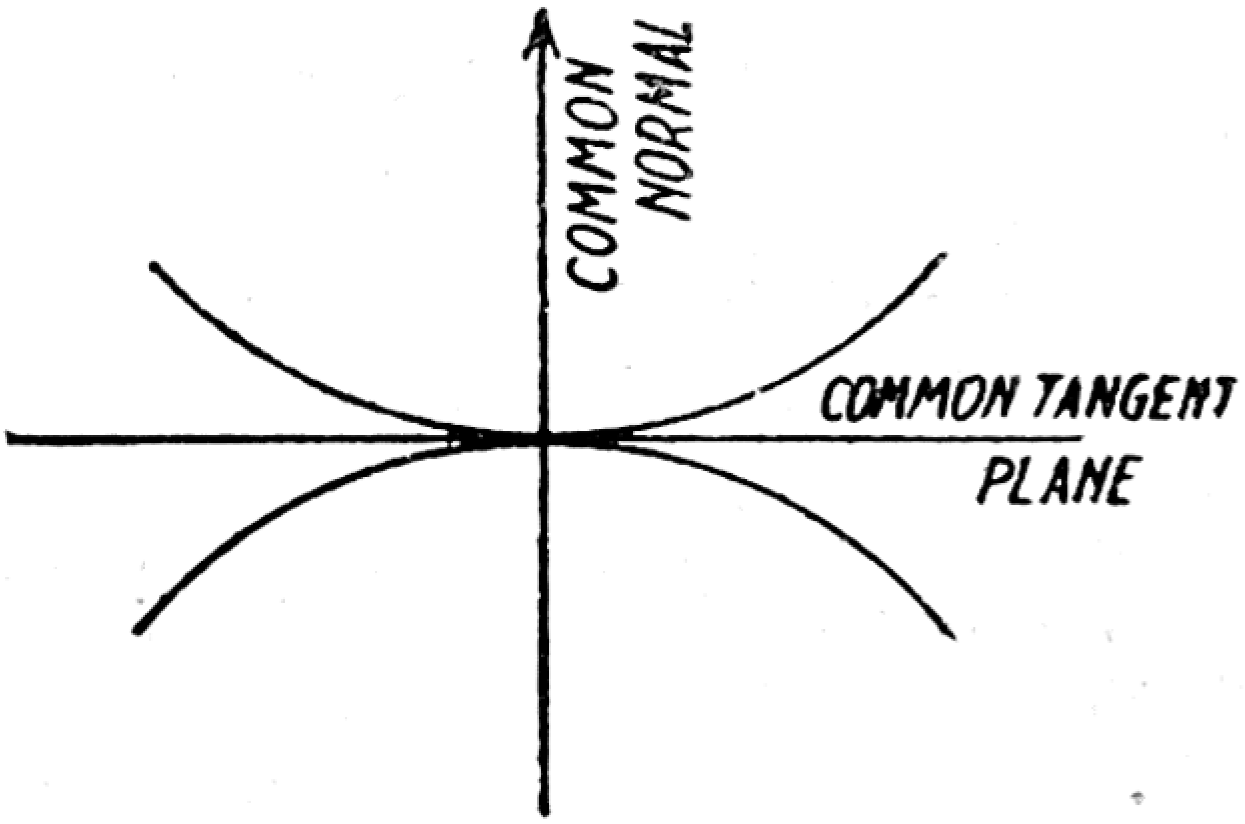
Surfaces are said to be **rough** when a tangential resistance is caused to the *sliding* (or *rolling*) of one over the other. *Surfaces are called* **smooth**, when there is no such tangential resistance, to the relative motion of one with respect to the other.

The difference between smooth and rough bodies may also be explained by reference to the nature of the reaction between such bodies. When two bodies are in contact and press each other, each will push the other with a certain amount of pressure. *If the bodies be smooth*, this pressure is entirely along the *common normal* to the (common) surface of contact. A *smooth surface, may, therefore, be defined* as a surface which can exert upon a body in contact with it a pressure only in a direction which is normal to the surface. Smooth bodies are incapable of offering resistance in any other than a normal direction. On the other hand, in the case of *rough surfaces in contact*, the mutual action usually consists of the normal reaction and an *additional tangential resistance* called *friction*, which acts in a direcion opposite to that in which the bodies have a tendency to relative motion.

Note 1.—The direction of the *common normal* along which the normal reaction acts may be found in the following manner, when contact between the surfaces is at a definite point.

(*i*) When the surfaces in contact are such that one is curved and the other plane, the common normal is perpendicular to the plane surface at the point of contact.





(*ii*) When both surfaces are curved, the common normal is perpendicular to the *common tangent plane* at the point of contact.

(*iii*) When both surfaces are plane, the common normal is perpendicular to both the planes.

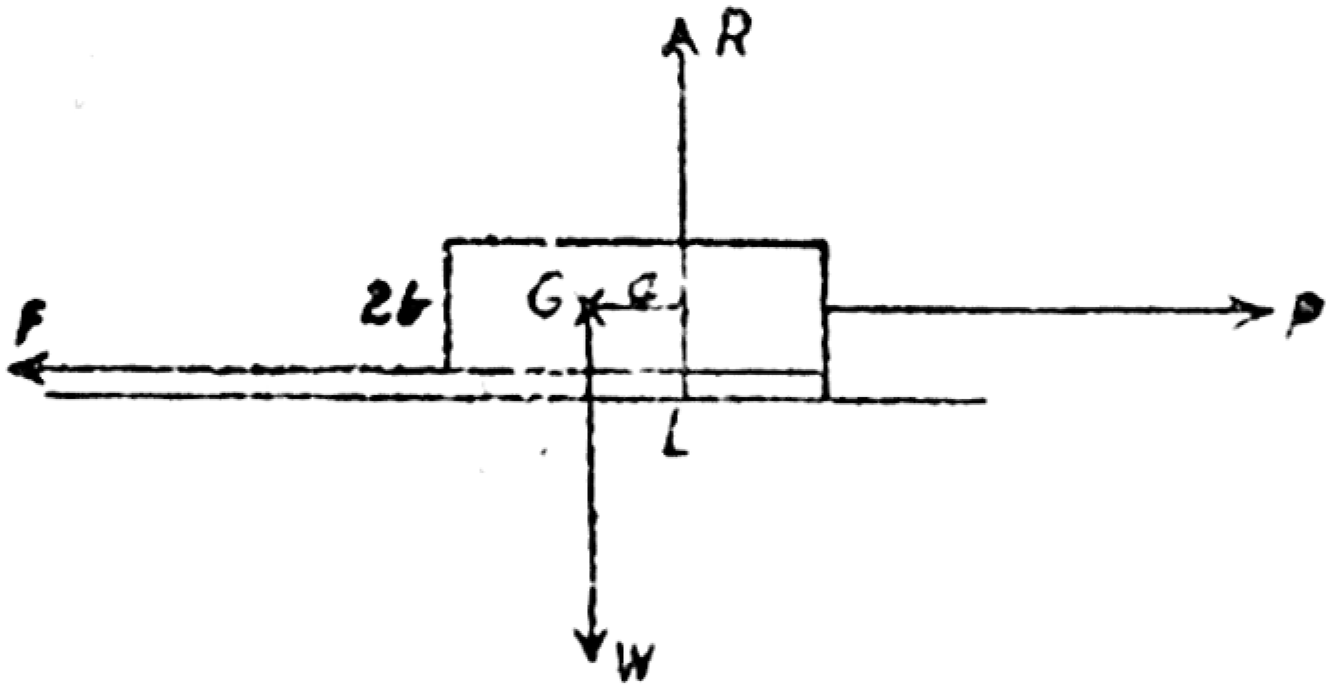
Note 2.—All bodies in nature are more or less rough. Bodies are never *perfectly smooth*; perfectly smooth bodies have existence only in the abstractions of Rational Statics and do not exist in nature. When we speak of smooth bodies, we merely treat them as such by neglecting their roughness.

129. Property of friction and force of friction. Definition. "If two bodies be in contact with one another, the property of the two bodies by virtue of which a resisting force is exerted between them at their point of contact to prevent one body sliding on the other is called **friction**, also the force exerted is called the **force of friction.**"

Note 1.—The resisting force which opposes the *rolling* of one body on another is called **rolling friction** or **resistance to rolling.** In this book we shall confine ourselves to the treatment of **sliding friction** only. The student will meet with Rolling Friction in Dynamics.

Note 2.—The term "friction" ordinarily stands for the "*force of friction*", but it is sometimes used to indicate the *property of friction*.

130. Consider a block of wood lying at rest on a table.



Let the block be pulled in the horizontal direction by a *small force* P, and we **observe** *that the block does not move.* The *following questions* naturally arise :—

(*i*) What kind of force is resisting its motion ?
(*ii*) What is the magnitude of this resisting force ?
(*iii*) In what direction is it acting ?

The reaction R of the table and the weight W of the block, being vertical forces, have no effect in the horizontal direction. They, therefore, cannot prevent the motion. *Since the block is at rest, there must be a force F opposing the pull P* and F = P in magnitude. The tangential resistance F is *the force of friction* exerted between the *rough bodies*, the block and the table.

We further **observe** that *as* the pull P is *gradually increased*, the frictional force *F increases with P*, so as to balance P at each instant, with the result that the block *does not move*. This state of things will continue till the variable pull, P, reaches a certain limit, which happens when the block *is just bordering on motion*. At this stage, the *force of friction has reached its maximum value*. When P *exceeds* this limit, F cannot increase further (having already reached its maximum) ; the equilibrium is broken and the block actually moves to the right. From this experiment we **infer** that

(*a*) So long as the block is at rest, the amount of friction (statical friction) developed depends on P ; and as P is increased, a gradually increased amount of friction is called into play to resist P and at each instant F = P ; statical friction is, thus, a *self adjusting* ($\therefore$ variable) force and the amount of friction exerted depends on the external forces applied.

(*b*) If P ceases to act, the frictional force F is zero; for, if it has a value other than zero, the body would move in the backward direction which is contrary to experience.

(*c*) The amount of (statical) friction developed in different cases varies from zero up to a certain *maximum value*, and is never more than that required to prevent motion.

(*d*) If $2b$ be the thickness of the block, the *conditions of equilibrium* are

$$R = W,$$
$$F = F;$$
$$W.a \not< b.P \text{ (moments about L).}$$

131. Statical (or non-limiting), dynamical and limiting friction.

(*a*) When one body in contact with another is in equilibrium but is *not bordering on motion*, the friction exerted is just sufficient to maintain equilibrium and is called *statical friction. The equilibrium* in this case, *is said to be non-limiting.*

(*b*) When one body is just on the point of sliding on another, the friction exerted attains its **maximum value** and is called *limiting friction; the equilibrium in this case is said to be limiting.*

(*c*) When motion ensues by one body sliding on another, the friction exerted between the bodies is called *dynamical friction.* It has been found that *dynamical friction is less than the limiting friction.*

132. The **following laws** govern the direction and magnitude of all the **three different kinds of friction**, namely,

(*i*) friction in non-limiting equilibrium.

(*ii*) friction in limiting equilibrium, and

(*iii*) friction in actual motion.

Law I, giving the direction of statical, dynamical and limiting friction.

Note. Friction always opposes motion or tendency to motion. When two bodies are in contact the direction of friction on any of them at its point of contact is opposite to the direction (*i*) in which this point of contact would begin to move if there were no friction, (*ii*) or in which it actually moves.

Law II, giving the magnitude of statical friction.

When there is equilibrium, the magnitude of the force of friction is just sufficient to prevent the body from moving.

Law III, giving the magnitude of limiting friction.

(*a*) *The magnitude of limiting friction always bears a constant ratio to the normal reaction.*

(*b*) *This constant ratio depends only on the substances of which the bodies are made.*

(*c*) *The limiting friction does not depend on the extent and the shape of the surfaces in contact.*

Law IV, giving the magnitude of dynamical friction.

When motion ensues by one body sliding on another, the friction is independent of the velocity; but the ratio of the friction to the normal reaction is slightly less than when the equilibrium is about to break.

Observation. These laws are not rigorously true but they express with a fair degree of accuracy the results of a large number of experiments. We shall, however, *assume* them as correct. It may be remarked that in Statics, we are not concerned with Law IV.

132. Co-efficient of friction. The ratio of the limiting friction to the normal reaction is called *coefficient of friction.*

When a rough body is on the point of sliding on another, the friction exerted bears a constant ratio to the normal reaction (Law III). In symbols, if F be the friction and R the normal reaction,

$$\frac{F}{R}=\mu=a \text{ constant},$$

i.e., $F=\mu R$ *when equilibrium is limiting.* μ is, therefore, different for different pairs of substances. Again, since friction attains its maximum value when it is limiting, **μR is the maximum value of friction.**

Note 1. $\frac{F}{R} < \mu$, *i. e.*, $F < \mu R$, if *equilibrium be non-limiting.*

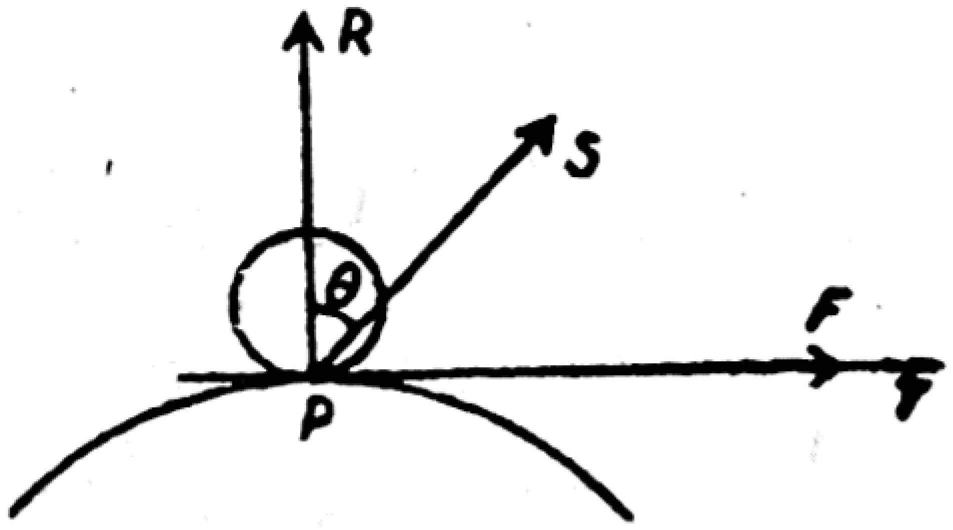
Note. 2. In limiting friction, F and R may both vary with the masses of the bodies even for the same pair of substances but the ratio $\frac{F}{R}$ is constant, being equal to μ.

133. Magnitudes of co-efficients of friction. Practically, no co-efficient is much greater than unity. In Rankine's Tables, the co-efficient for damp clay on damp clay is given as 1 and for shingle on gravel, it is at most 1·11. But most of the co-efficients are less than $\frac{1}{2}$. Thus μ *may ordinailry be taken as less than 1.*

The co-efficients and angles of friction for some other pairs of substances will be seen from the following table :—

Substances		μ	λ
Wood on wood	... Dry	·25 to ·5	14° to $26\frac{1}{2}^\circ$
,, ,, ,,	... Soaped	·04 to ·2	2° to $11\frac{1}{2}^\circ$
Metals on metals	... Dry	·15 to ·2	$8\frac{1}{2}^\circ$ to $11\frac{1}{2}^\circ$
,, ,, ,,	... Wet	·3	$16\frac{1}{2}^\circ$
Leather on metals	... Dry	·56	$29\frac{1}{2}^\circ$
,, ,, ,,	... Wet	·36	20°
,, ,, ,,	... Oily	·15	$8\frac{1}{2}^\circ$

134. Total Resistance. Let a body be *resting in equilibrium* on a rough surface AB, under the action of forces in the plane of the paper. Let R be the normal reaction and F the force of friction acting along the tangent PT in a direction opposite to that in which motion *would have ensued if there were no friction.* The total reaction of the surface consists of two forces R, and F, and it is found convenient to combine them



into a single force. The resultant of R and F (denoted by S in the figure) is a force which is called the **total resistance** of the surface. The angle θ between S and the normal is

$$\text{given by } \tan\theta = \frac{F}{R} = \mu.$$

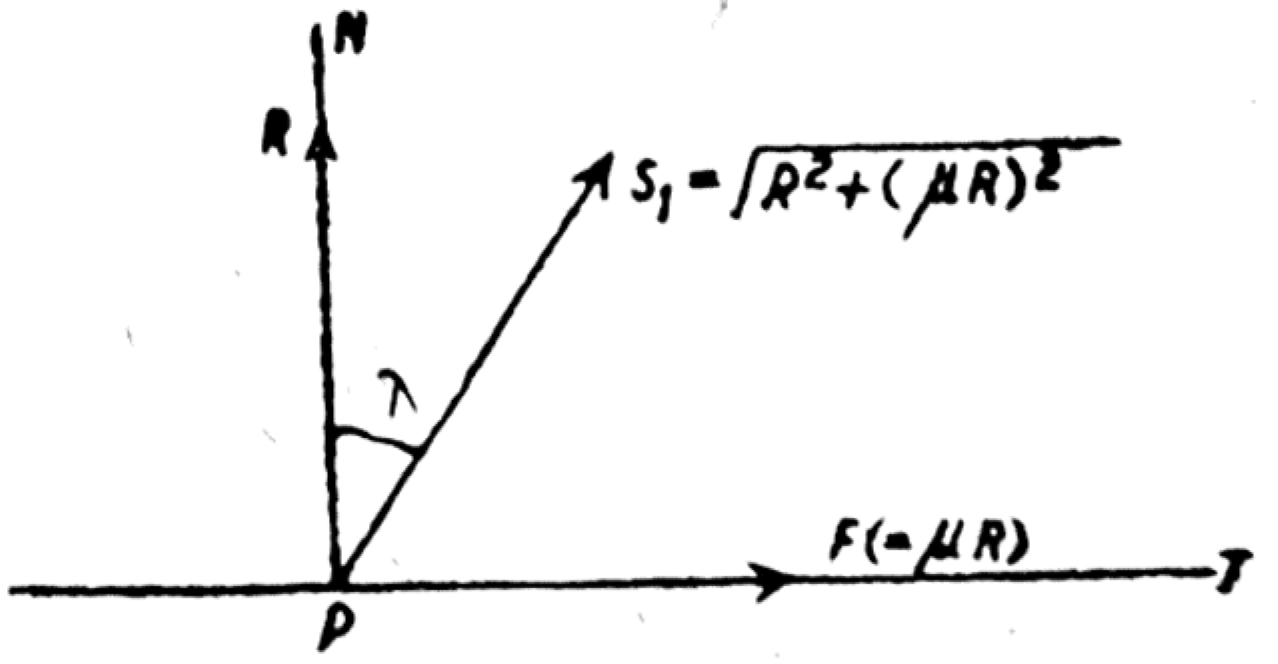
As F increases, the angle θ increases, and it will be the greatest when the forces of friction bears the greatest ratio to the normal reaction, which happens when the body is just on the point of sliding along the surface and this greatest ratio, as we have already seen, is called the *co-efficient of friction* μ.

Note.—The *total resistance* is sometimes called *resultant reaction* of the surface.

135. Angle of friction : Definition. When a body acted on by forces is in equilibrium on a rough surface but *slipping is just about to take place* so that the friction exerted is the limiting friction, the angle which the resultant reaction of the surface makes with normal to the surface at the point of contact is called the *angle of friction.*

To show that the tangent of the angle of friction is equal to the co-efficient of friction.

Let F be the *limiting friction* and R the normal reaction.



Then $F = \mu R$, (Law III) μ being the co-efficient of friction.

Let S_1 be the resultant reaction making an angle λ with R ; then λ is the angle of friction.

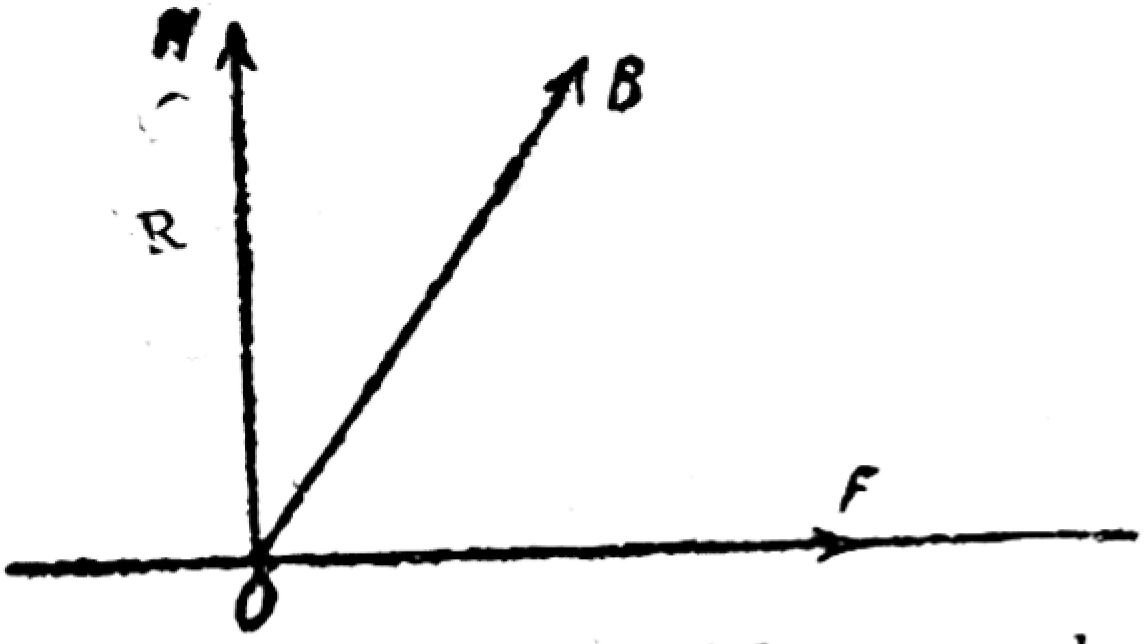
R and μR are the resolved parts of S_1 along PN and PT.

$\therefore \quad S_1 \sin\lambda = \mu R$

and $S_1 \cos\lambda = R$

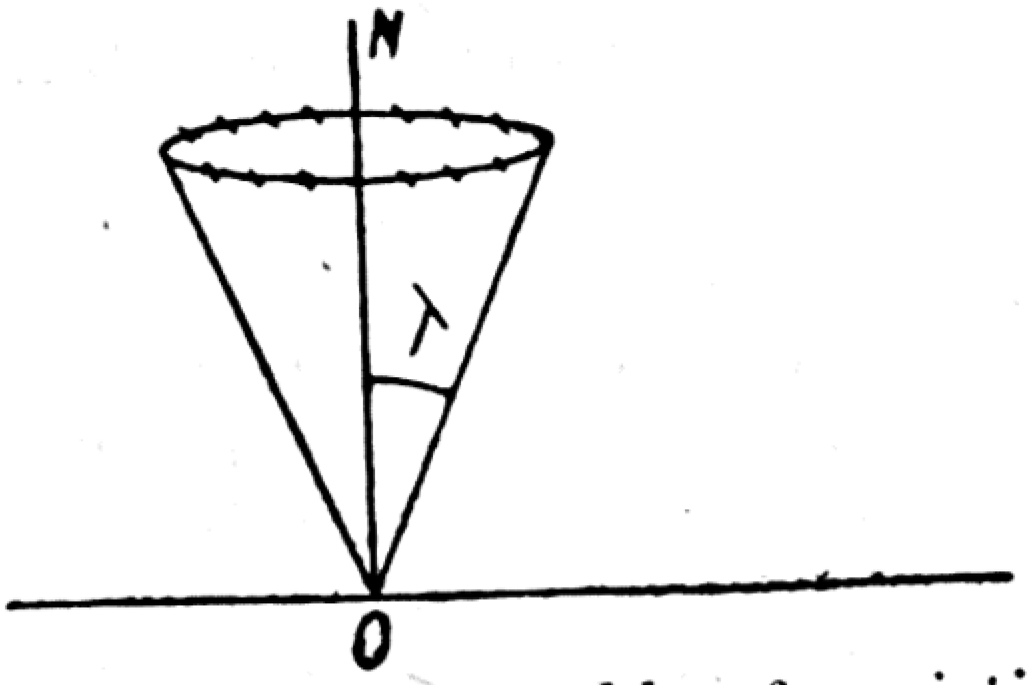
$\therefore$ by division $\tan\lambda = \mu$.

136. Since the magnitude of the frictional force varies from zero to its maximum value μR, it is evident that the angle between the total resistance (resultant reaction) and the normal reaction varies from zero to its maximum value $\lambda (= \tan^{-1}\mu)$.



Thus, if a body be in equilibrium in contact with a rough surface at O, and we draw the normal ON and the straight line OB making an angle λ with ON as shown, the resultant reaction may have any direction between ON and OB but the line of action of the resultant reaction cannot be outside the angle NOB. Now, the body in equilibrium may be pulled *in any of the infinite directions* passing through O in the tangent plane and the friction will act so as to oppose motion (tendency to motion). If we describe a cone about ON with semi-vertical angle equal to λ (called the *cone of friction*), *the resultant reaction may have any direction inside this cone or on its surface but the line of action of this resultant cannot be outside the cone.*

By increasing R, we can increase F so that we can make F of any magnitude we please, but we cannot bring the direction of the resultant reaction ($= \sqrt{F^2 + R^2}$) outside the cone.



Definition. The cone described with its vertex at the point of contact of two rough surfaces and having the common normal as axis and the angle of friction as the semi-vertical angle is called the **cone of friction.**

Note.—Assuming that the surfaces are capable of resisting penetration to any extent, it is clear from what has been said above that the total resistance *can assume any magnitude but there is a limitation to the direction* of the total resistance, its inclination with the normal never exceeding λ " *The total resistance of a rough curve or surface, though unrestricted in magnitudea can never make with the normal an angle greater than the angle of friction corresponding to the two bodies in contact.* Within this limit, the total resistance can assume any magnitude and direction at the circumstance of the case demand. **(Minchin)**

137. Active and passive Forces. We can distinguish between *two kinds of forces* by the mode of their action : (*i*) **moving or active forces** which tend to produce motion and (*ii*) **passive resisting forces** or **passive resistances** which tend to resist or change motion. All *resistance* (*e.g.*, resistance offered by air or water to a body in motion) and *reaction and friction are passive forces,* as they have no tendency to *produce* motion but have a tendency to *modify or resist* motion. They come into existence on account of the action of other moving forces for the purpose of preventing some motion.

138. Friction is a passive resistance. Friction, as we have seen, is a passive force and comes into existence to oppose motion likely to be caused by the moving forces acting on the body, and to destroy the effect of these forces. It sole function is to prevent the relative motion of the body. *Resisting forces are determined by the moving forces against which they act.*

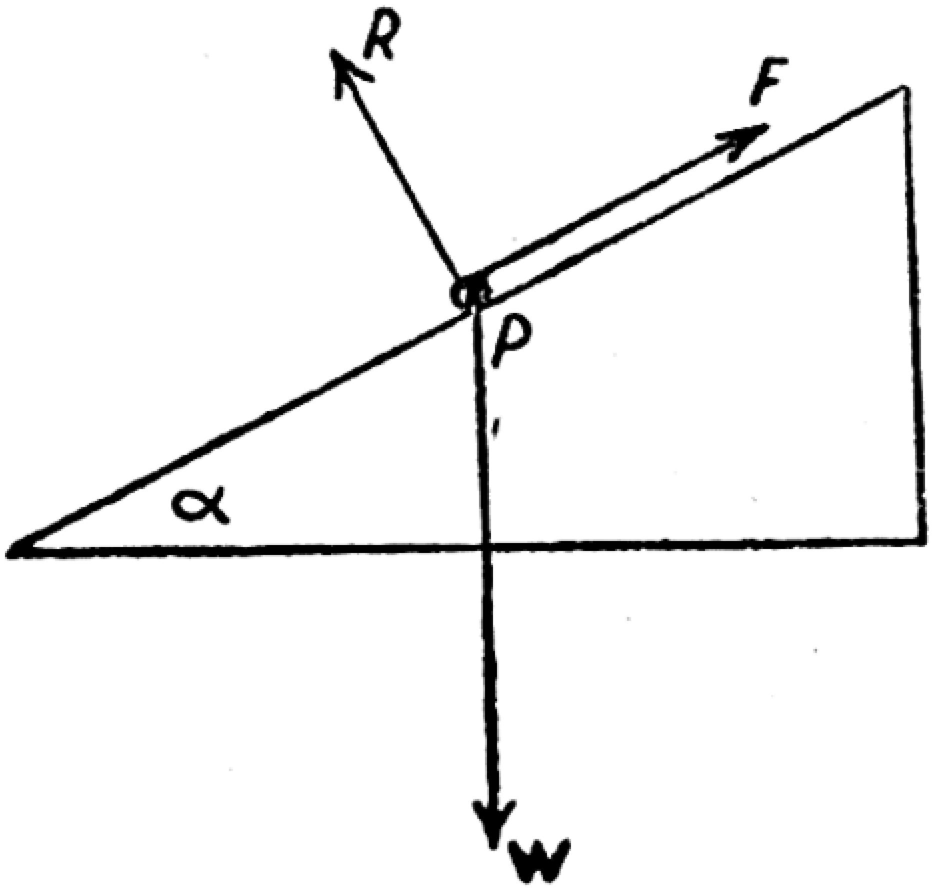
139. Law of passive forces.

"*If passive resistances can give equilibrium they will.*"

When friction acts on a body, this statement means that " the total resistance (resultant of R and F) which acts on a body at any point of a rough surface, will, if possible, assume such a magnitude and direction as will preserve equilibrium " by balancing the resultant of other moving forces acting on the body.

" A *passive force* being entirely due to the tendency to motion caused by *active forces,* only resists such tendency : its direction, therefore, is always directly opposite to the motion resisted and its magnitude never exceeds the minimum required for preserving equilibrium and is, if possible, equal to this minimum. **(Greaves)**

The following *criterion for the existence of friction* follows at once from this law. When two rough bodies are in contact, if the forces acting on the body are arranged in such a manner that there is no tendency to slip at their point of contact, no friction would be exerted, *i.e.*, *the magnitude of friction would be zero.* Generally, in the case of all passive resistances, *if there is no tendency to displacement which a passive resistance is required to prevent this force will not come into play.*"



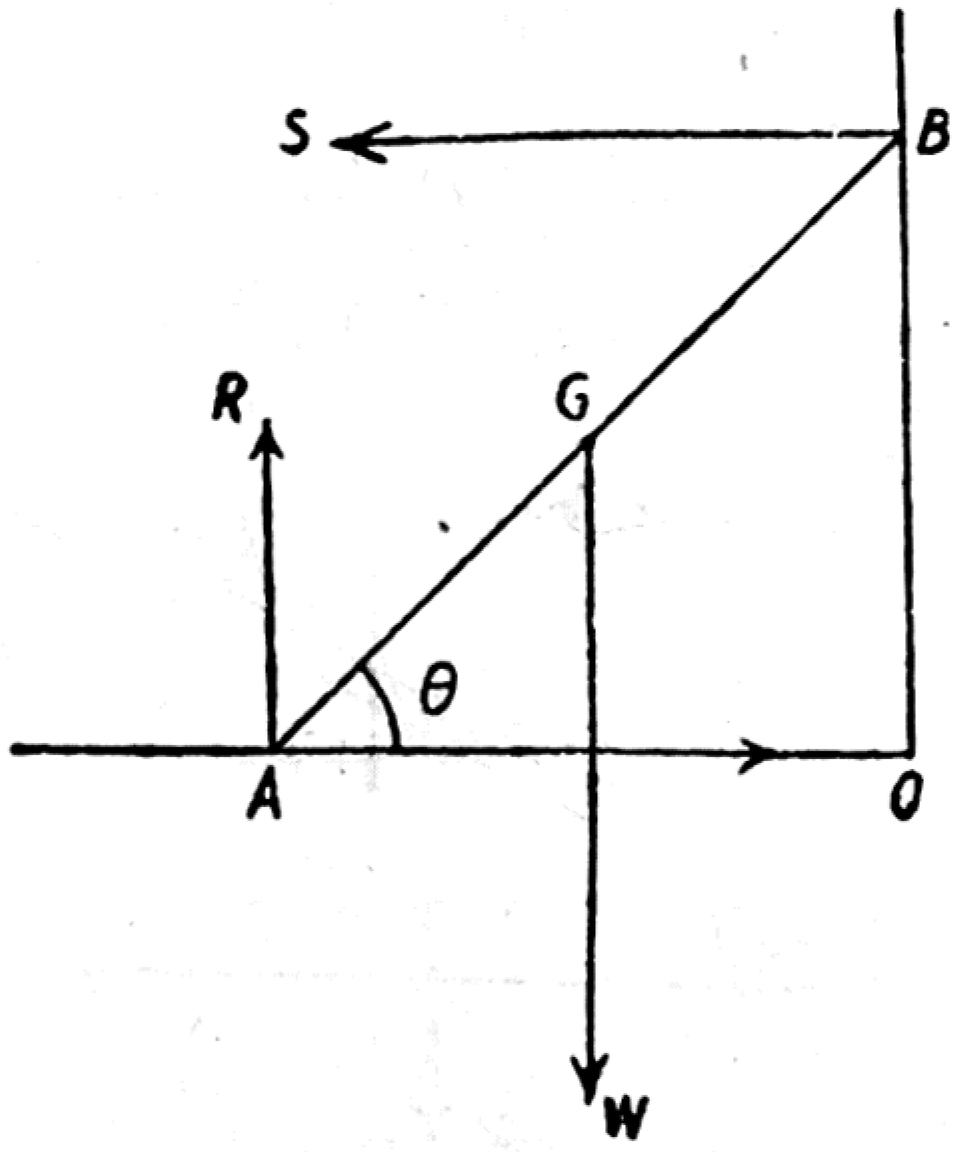
Ex. 1. *A particle of weight W is at rest on the inclined plane of angle α; find the friction that is called into play and the reaction of the plane.*

Let R be the normal reaction. If there is no friction, the particle would slide down the plane Therefore friction must act up the plane. Let it be F.

Resolving along and perpedicular to the plane,

$$F = W \sin \alpha,$$
$$\text{and } R = W \cos \alpha.$$

Cor. $F = R \tan \alpha$, so that equilibrium is possible only if $\tan \alpha \leqslant \mu$ $(= \tan \lambda)$, *i.e.*, if $\alpha \leqslant \lambda$.



Ex. 2 *A uniform ladder, of Weight W is in equilibrium with one end resting on the rough ground and the other end against a smooth vertical wall. If the inclination of the ladder to the horizontal is θ find the friction and reaction at the wall.*

Let AB be the ladder, and G its centre of gravity; let R and S be the normal reactions at A and B respectively.

If there were no friction at A, the three forces R, W and S cannot keep equilibrium (because they are neither parallel nor concurrent) and end A of the ladder would slip away from the wall so that friction must act at A along AO; let it be F.

Resolving horizontally and vertically, we have

$$S = F \text{ and } R = W.$$

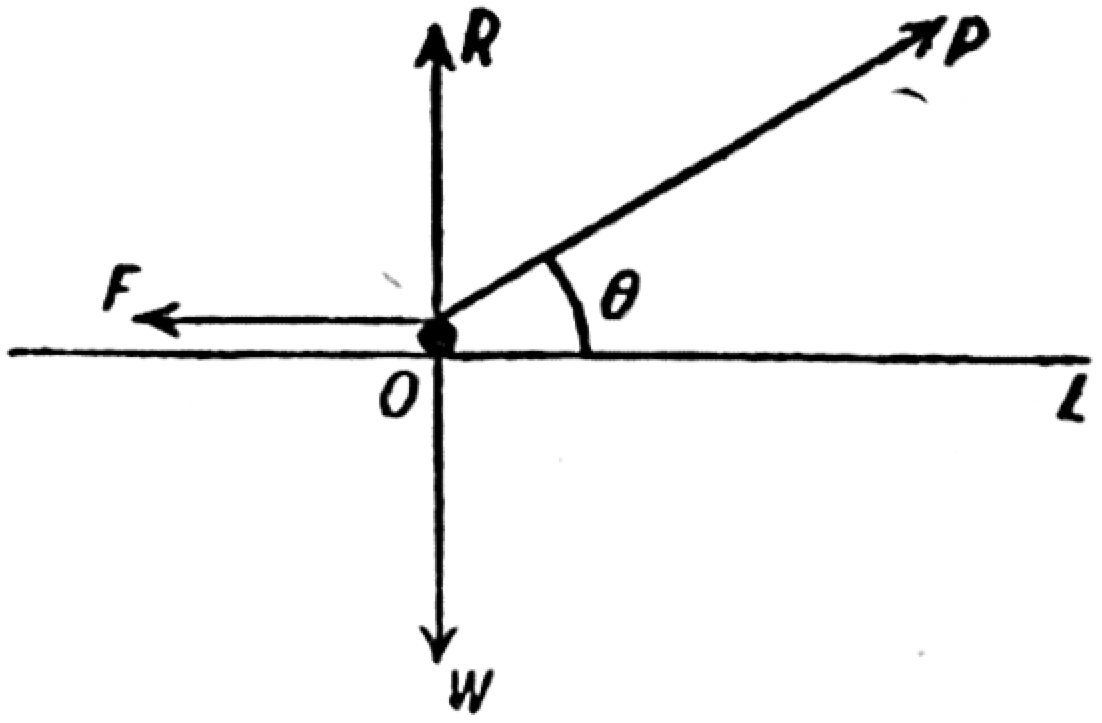
Taking moments about A

$$W.AG. \cos\theta = S.\ AB \sin\theta$$

$$\text{or } S = \frac{W}{2} \cot\theta.$$

$$\text{Hence } F = \frac{W}{2} \cot\theta.$$

Ex. 3. *A particle of weight W lying on a rough table, is acted upon by a force P making an angle θ with the horizontal. If the particle is at rest, find the friction that is called into play and the reaction of the table.*



If there were no friction, the particle would move along OL because the only force in that direction would be P cos θ. Hence friction must act along LO. Let it be F.

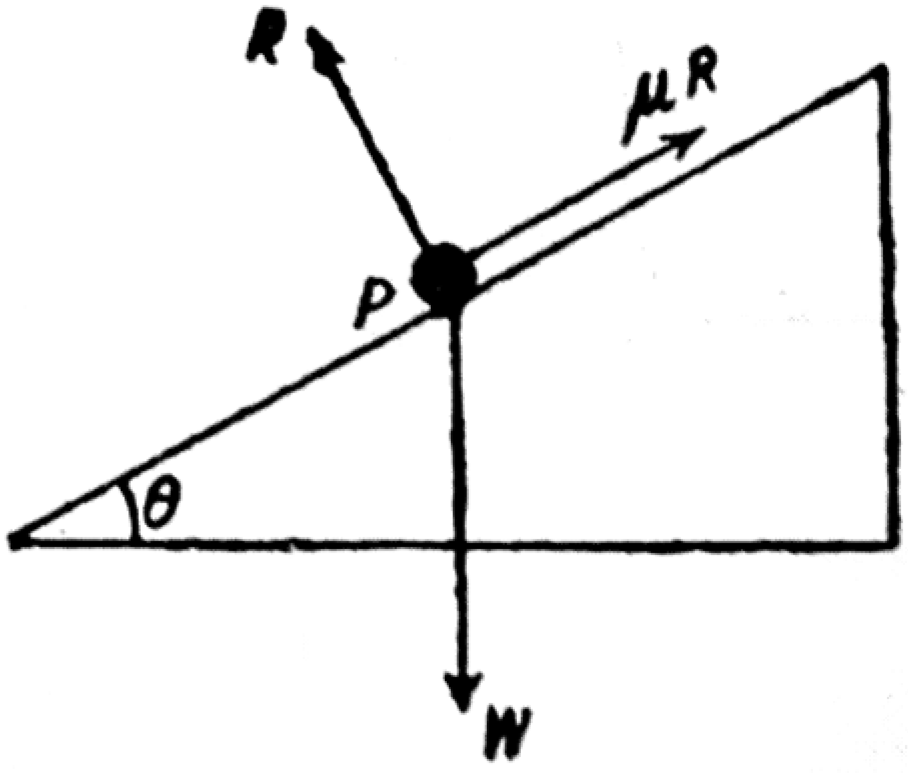
Resolving, we get

$$R + P \sin\theta = W$$

$$\text{and } F = P \cos\theta.$$

$$\therefore\ R = W - P \sin\theta \text{ and } F = P \cos\theta.$$

Ex. 4. *A particle is resting on a rough inclined plane, the co-efficient of friction between the two being μ. If the inclination of the plane is gradually increased, find when the particle is on the point of sliding down.*



Let the particle be on the point of sliding down when the angle of the plane is θ.

If R be the normal reaction, and F the friction acting upwards, we must have $F = \mu R$ because the equilibrium is limiting.

Resolving along and perpendicular to the plane,

$$W \sin\theta = \mu R$$

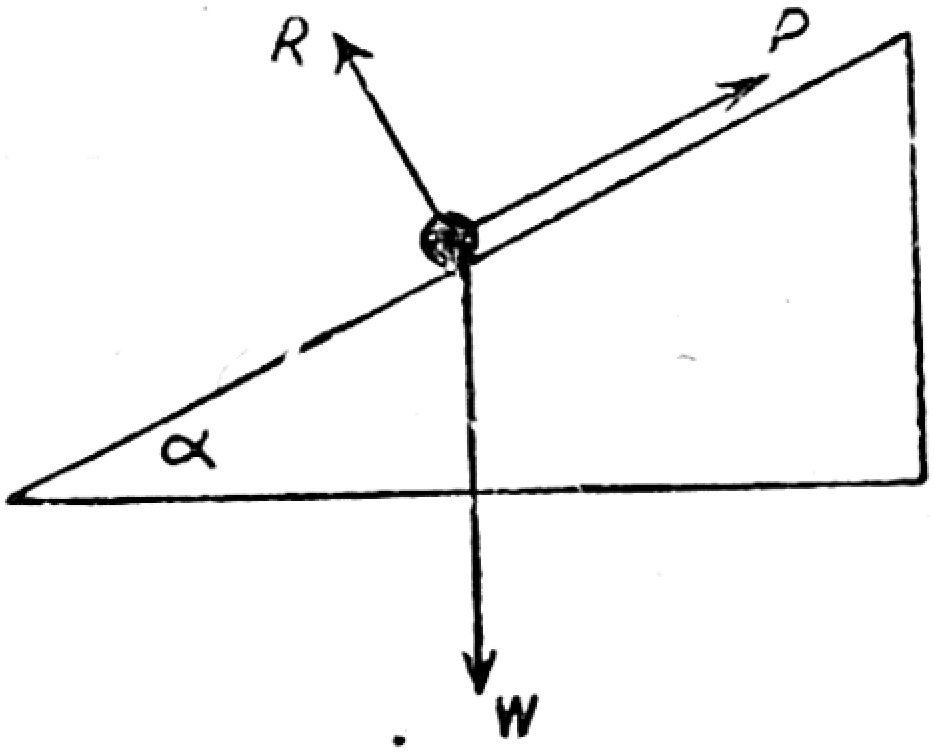
$$\text{and } W \cos\theta = R.$$

Dividing, we get $\tan\theta=\mu=\tan\lambda$ or $\theta=\lambda$.

So that the particle is on the point of sliding down if the angle of the plane is equal to the angle of friction between the particle and plane.

It may be noticed that this example gives a practical way of finding out the co-efficient of friction between two substances.

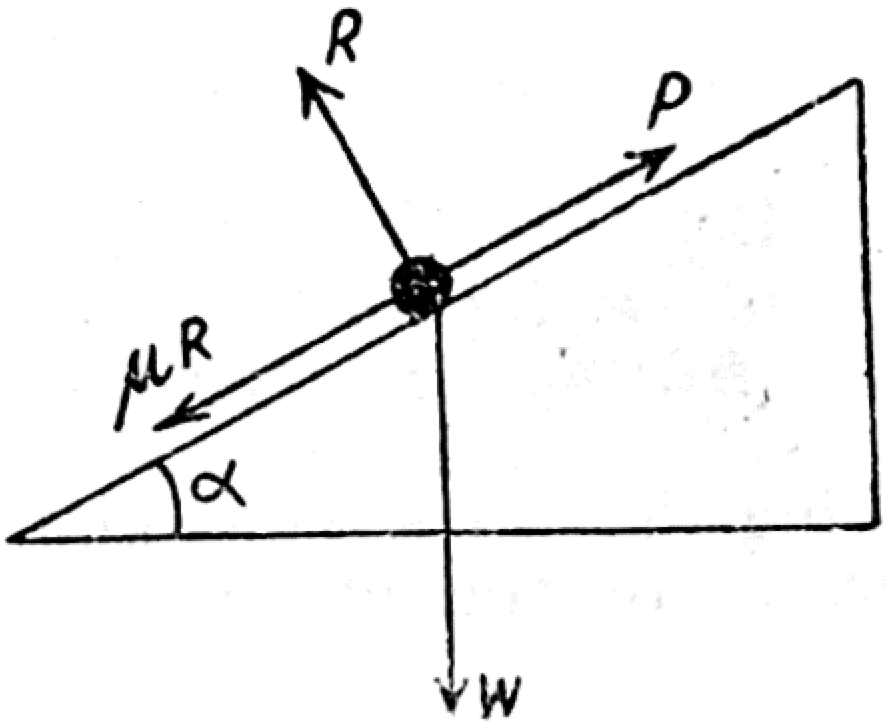
Ex. 5. *A particle of weight W is resting on a rough inclined plane of inclinate α, being acted upon by a force P up the plane. Find P so that no friction may be called into play.*



It is obvious that if the three forces R, P and W be in equilibrium, no friction is called into play.

Hence $P=W\sin\alpha$.

Ex. 6. *A body of weight W is at rest on a rough inclined plane, of inclination α greater than the angle of friction, being acted upon by a force parallel to the plane and along a line of greatest slope; to find the limits between which the force must lie.*



Let α be the inclination of the plane, and R the normal reaction.

Case 1. Let the body be on the point of sliding *up the plane,* so that friction μR acts down the plane.

Let P be the force required to keep the body at rest.

Resolving along and perpendicular to the plane, we have,

$$P-\mu R=W\sin\alpha \quad (i)$$

$$R=W\cos\alpha \quad (ii)$$

$$P=\mu R+W\sin\alpha=W(\sin\alpha+\mu\cos\alpha)$$

so that if the force be $> W(\sin\alpha+\mu\cos\alpha)$, the body shall actually move up the plane.

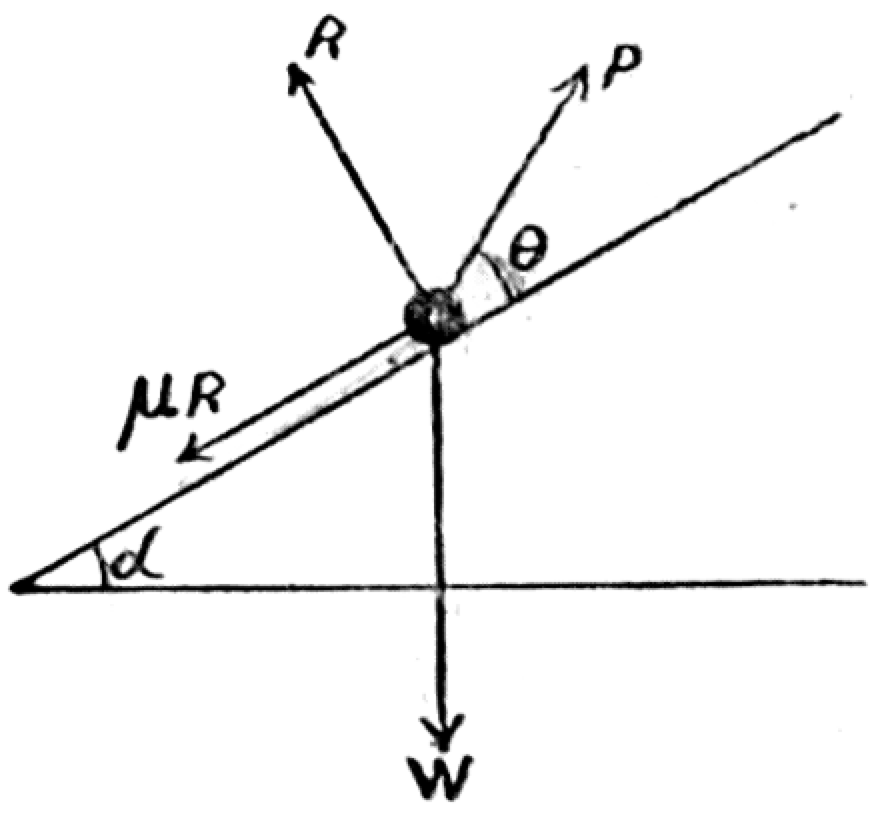
Case 2. Let the body be on the point of sliding *down the plane,* so that friction acts up the plane. In this case we have

$$P_1+\mu R_1=W\sin\alpha$$
$$R_1=W\cos\alpha$$
$$\therefore\ P_1=W(\sin\alpha-\mu\cos\alpha)$$

so that if the force $< W(\sin\alpha-\mu\cos\alpha)$, the body shall slide down the plane. Hence the given force must lie between $W(\sin\alpha+\mu\cos\alpha)$ and $W(\sin\alpha-\mu\cos\alpha)$.

It may be observed that the result for case (2) is obtainable from that of case (1) by changing the sign of μ only.

Ex. 7. *A body of weight W rests on a rough inclined plane of inclination α to the horizon, being acted upon by a force making an angle θ with the plane. To find the limits between which the force must lie. Find also the magnitude and direction of the least force required to drag the body up the inclined plane.*



Let P be the force acting at angle θ with the plane, R the normal reaction and W the weight of the body.

Case (*i*) Let the body be on the point of *moving up the plane so that the friction acts down the plane.*

Resolving the forces along and perpendicular to the plane, we have

$$P\cos\theta=\mu R+W\sin\alpha\ \ldots\ldots(1)$$
$$P\sin\theta+R=W\cos\alpha\ \ldots\ldots(2)$$

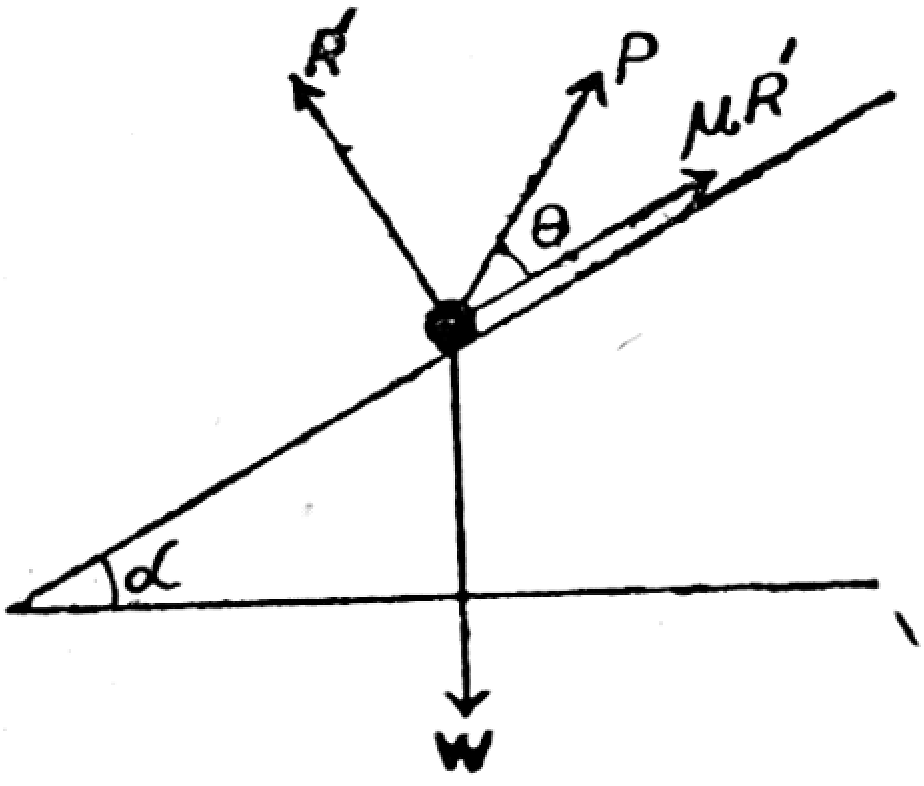
To eliminate R, we substitute from (2) the value of R in (1); we get

$$P\cos\theta=\mu(W\cos\alpha-P\sin\theta)+W\sin\alpha,\ \text{whence}$$
$$P(\cos\theta+\mu\sin\theta)=W.(\mu\cos\alpha+\sin\alpha).$$

$$\therefore\ P=W.\frac{\mu\cos\alpha+\alpha\sin}{\cos\theta+\mu\sin\theta}$$

$$= W.\frac{\tan\lambda\cos\alpha+\sin\alpha}{\cos\theta+\tan\lambda\sin\theta}$$

$$= W.\frac{\sin(\alpha+\lambda)}{\cos(\theta-\lambda)} = P_1.$$



Case (*ii*) Let the body be on the point of *moving down the plane* so that the friction *acts up the plane.*

In this case, the conditions of equilibrium are

$P \cos\theta + \mu R' = W \sin\alpha$ (1)

$P \sin\theta + R' = W \cos\alpha$ (2)

Multiply (2) by μ and subtract; we eliminate R' and get

$P(\cos\theta - \mu\sin\theta) = W(\sin\alpha - \mu\cos\alpha),$

whence,

$$P = W\frac{\sin\alpha-\mu\cos\alpha}{\cos\theta-\mu\sin\theta}$$

$$= W.\frac{\sin(\alpha-\lambda)}{\cos(\theta+\lambda)} = P_2 \text{ (putting } \mu=\tan\lambda).$$

P_1 and P_2 are the limiting values of the force if the body is to remain in equilibrium on the inclined plane.

Note.—If the force lies between P_1 and P_2 *the body remains in equilibrium but equilibrium is non-limiting* so that the body is not on the point of moving either up the plane or down the plane.

Case (*iii*) To find the direction and magnitude of the least force to drag the body up the plane, we have, from case (*i*) [*P. U. 1944*]

$$P = W\frac{\sin(\alpha+\lambda)}{\cos(\theta-\lambda)}. \quad ...(A)$$

P is a function of θ, α and λ being constants. We have to answer the question: "For what value θ, P is least?"

Evidently P is least when the denominator $\cos(\theta-\lambda)$ is greatest, *i.e.*, equal to unity. This happens when $\theta-\lambda=0$, *i.e.*, when $\theta=\lambda$.

Hence the required force P makes with the plane the angle of friction.

In this case, we have from (A)

$$P = W \sin(\alpha + \lambda).$$

Exercises XVI

1. A body of weight 80 lbs. rests on a rough horizontal plane while a force of 20 lbs. is acting on it in a direction making an angle of 60° with the horizontal. Find the force of friction that is called into play.

2. A uniform ladder, of weight 36 lbs., is in equilibrium, with one end resting on the rough floor and the other end against a smooth vertical wall. If the inclination of the ladder to the horizontal is 60°, find the friction.

3. A particle, of weight 45 lbs. is at rest on an inclined plane of inclination 30°, find the friction that is called into play and the reaction of the plane.

4. A body of 40 lbs. resting on a rough horizontal table is about to slide when pulled by a horizontal force of 20 lbs. Find the co-efficient of friction between the table and the body.

5. A force of 30 lbs. acting at an angle of 30° with the horizontal is about to drag a body of weight 60 lbs. lying on the the floor. Find the co-efficient of friction.

6. Show that the least force which will move a weight W along a rough horizontal plane is $W \sin \lambda$.

7. A heavy body is supported on a rough inclined plane of angle α, by a force inclined to the plane at an angle θ. Find the least and the greatest value of the force by which equilibrium is maintained.

8. Show that the force required to move a body up a rough inclined plane will be least when it is applied in a direction making with the inclined plane an angle equal to the angle of friction.

9. Two equally rough bodies, of weights W_1 and W_2, $(W_2 > W_1)$ on a rough inclined plane are connected by a string which passes round a fixed smooth small pulley in the plane; find the greatest inclination of the plane consistent with the equilibrium of the two bodies.

10. Two inclined planes have a common vertex and a string passing over a small smooth pulley at the vertex, support two equal weights. If one of the planes be rough (of inclination β) and the other smooth (of inclination α), find the relation between α and β when the weight on the rough plane is on the point of moving down the plane.

11. A paritcle is placed on the outside of a rough sphere, the co-efficient of friction being μ, show that it will be on the point of motion when the radius from it to the centre makes an angle $\tan^{-1}\mu$ with the vertical.

12. Find the work done in dragging a body of weight W, up a rough inclined plane of length l and angle α.

[*Hint.* From Ex. 6. P. 238 case (*i*) the force P which would just move the body up the plane is $W(\sin\alpha+\mu\cos\alpha)$. Hence work done in dragging it is, $W(\sin\alpha+\mu\cos\alpha)l$].

Note : From this example we deduce the following :—

The work done in drawing a body up a rough inclined plane is equal to the work done in drawing a body along a rough inclined plane equal to the base "b" of the plane, together with the work done in lifting the body through the height "h" of the plane.

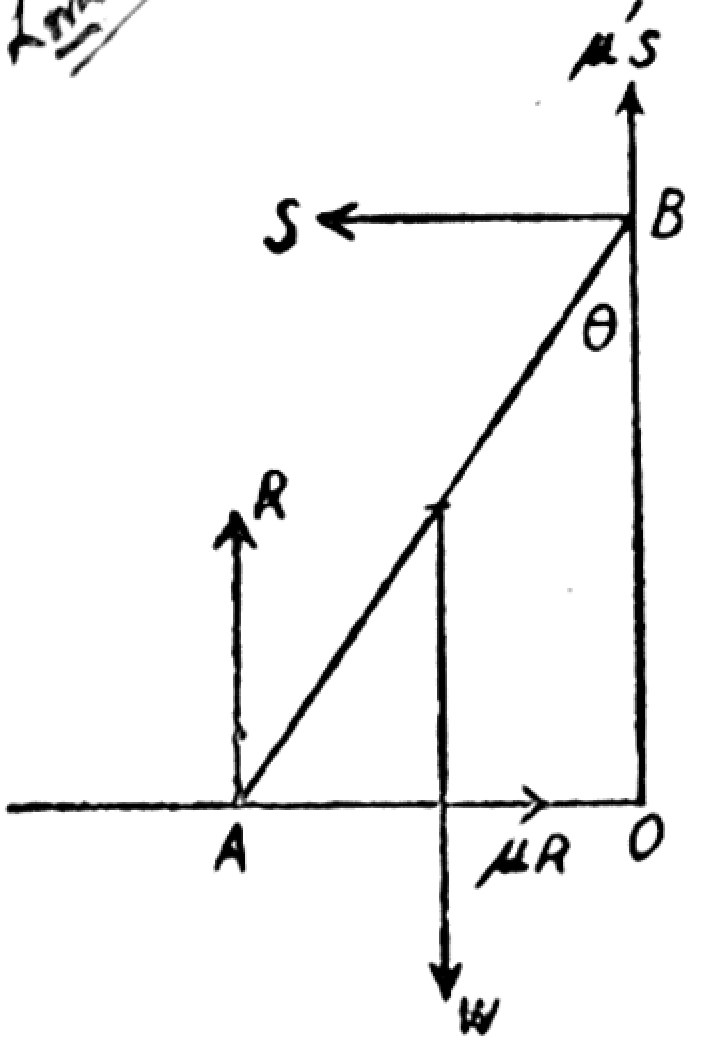
$$\text{The work done} = W(l\sin\alpha+\mu . l\cos\alpha) = W(h+\mu b)$$
$$= Wh+\mu Wb.$$

But (*i*) μWb = Work done in drawing the body along the horizontal plane through b.

and (*ii*) Wh = Work done in lifting the body up a height h.

140. We shall now solve a few **more problems** involving friction.

Ex. 1. *A uniform ladder is resting on a rough ground and leaning against a rough wall : find the greatest angle at which the ladder can be inclined to the vertical without slipping, given the co-efficients of friction for the ground and the wall are μ and μ' respectively.*



Let θ be the inclination of the ladder to the vertical and $2l$ its length.

Resolving horizontally and vertically we have

$$S = \mu R \qquad \ldots(i)$$

and $W = R + \mu' S$.

Also taking moments about B, we have

$$Wl \sin\theta + \mu R\, 2l \cos\theta + R\, 2l \sin\theta.$$

or $W \sin\theta = 2\,R\,(\sin\theta - \mu \cos\theta) \ldots\ldots(iii)$

From (i) and (ii), we have

$$W = R + \mu'\mu R = R(1 + \mu\mu') \qquad \ldots\ldots(iv)$$

From (iii) and (iv) by division, we get

$$\sin\theta = \frac{2(\sin\theta - \mu\cos\theta)}{1 + \mu\mu'}$$

or $2\mu \cos\theta = (1 - \mu\mu') \sin\theta$

or $\quad \tan\theta = \dfrac{2\mu}{1 - \mu\mu'}$.

Ex. 2. *Two equal uniform rods AC, CB are freely jointed at C and rest in a vertical plane with ends A and B on a rough horizontal plane. Show that in limiting equilibrium angle ABC is $2 \tan^{-1}(2\mu)$, μ being the co-efficient of friction.*

There being symmetry in the figure, the reactions at C shall be horizontal and the normal reactions at A and B shall be equal.

Resolving for the whole system.

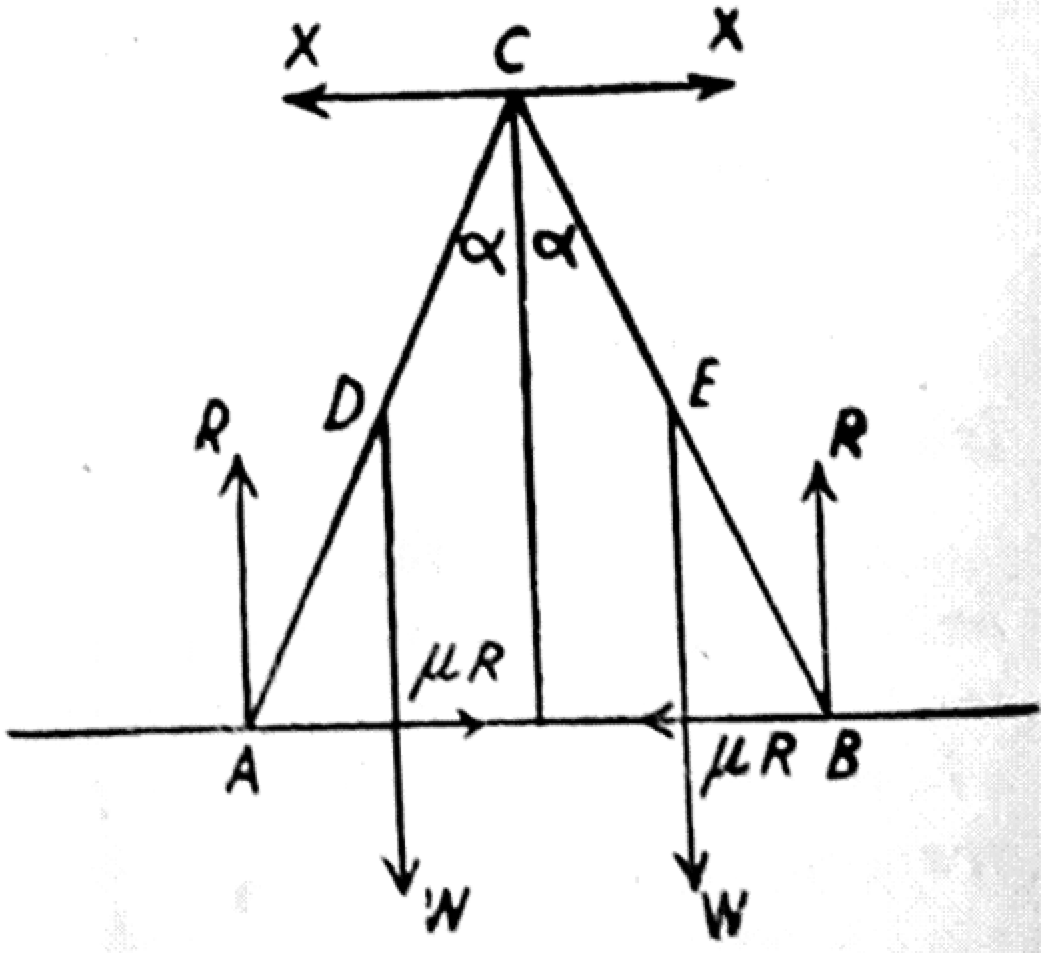
$$2W = 2R \qquad \ldots\ldots(i)$$

Resolving for the rod AC only,

$\mu R = X, \quad R = W$ and

$\therefore\ X = \mu W$.

Taking moments about C for the equilibrium of the rod AC only, we have



$$W.CD \sin \alpha + \mu R.AC \cos \alpha = R.AC \sin \alpha$$

or $\sin \alpha + 2\mu \cos\alpha = 2 \sin \alpha.$

$\therefore \quad \tan \alpha = 2\mu$

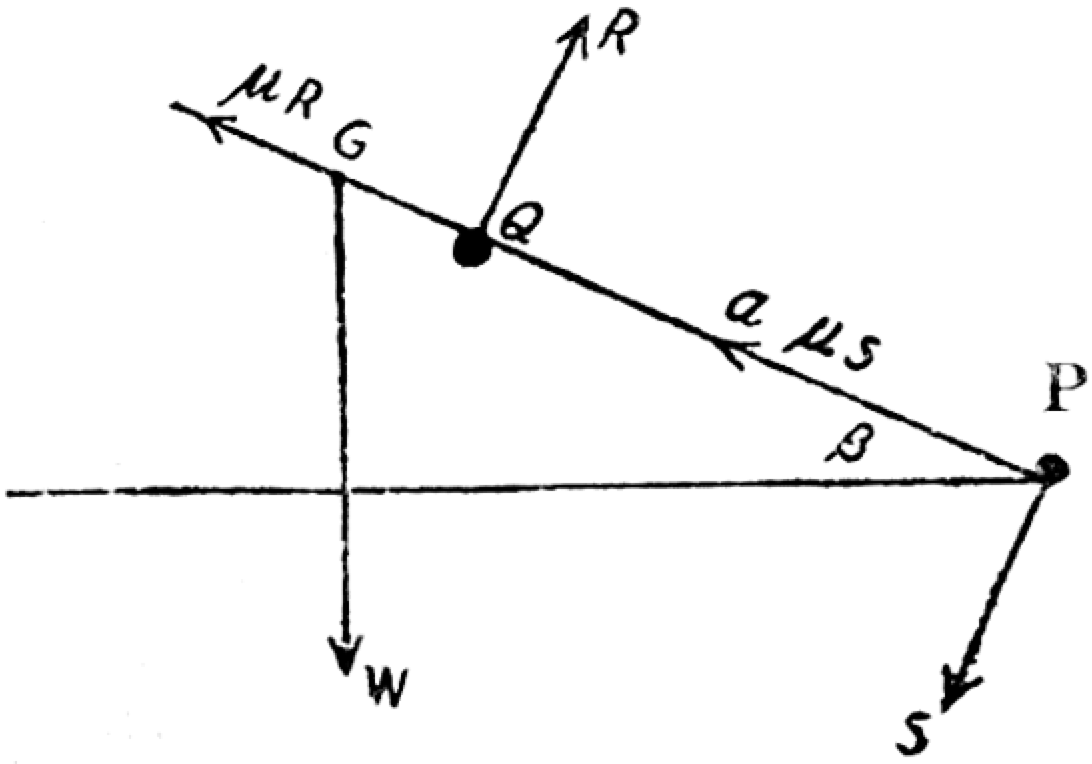
or $\quad \alpha = \tan^{-1} (2\mu).$

Hence angle ACB is $2 \tan^{-1} (2\mu).$

It may be observed that it is not absolutely necessary to make use of the symmetry of the figure. The problem can still be solved if the reaction at B is taken as R′ and two more equal and opposite vertical reactions, each equal to Y, are introduced at C. But it is obvious that the solution will be a little more lengthy.

Ex. 3. *A thin uniform rod passes over one peg and under another, the co-efficient of friction between each peg and the rod being* μ. *The distance between the pegs is* α, *and the line joining them makes an angle* β *with the horizontal. Show that equilibrium is not possible unless the length of the rod is greater than*

$$\frac{a}{\mu}(\mu + \tan\beta).$$



The rod is of minimum length if the lower end is at P below the lower peg, C. G. of the rod being above Q, the other peg. We want 2 P.G.

Resolving, we get

$$W + S \cos \beta = R \cos \beta + \mu R \sin \beta + \mu S \sin \beta \qquad \ldots\ldots(i)$$

$$R \sin \beta = S \sin \beta + \mu S \cos \beta + \mu R \cos \beta \qquad \ldots\ldots(ii)$$

Taking moments about P, $W.PG \cos \beta = Ra$

$$\therefore \quad PG = \frac{a}{\cos \beta} \cdot \frac{R}{W}.$$

From (*i*) and (*ii*) we get

$$S (\cos \beta - \mu \sin \beta) = R \cos \beta + \mu R \sin \beta - W$$

$$\text{and } S (\sin \beta + \mu \cos \beta) = R \sin \beta - \mu R \cos \beta$$

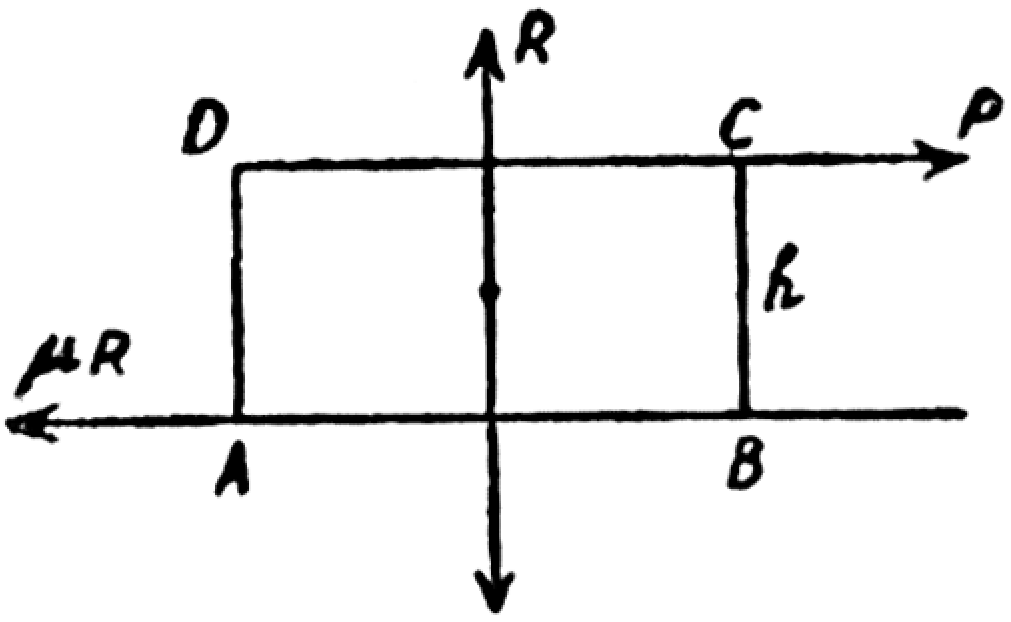
$$\therefore \quad \frac{\cos\beta-\mu\sin\beta}{\sin\beta+\mu\cos\beta}=\frac{R(\cos\beta+\mu\sin\beta)-W}{R(\sin\beta-\mu\cos\beta)}$$

$$=\frac{\cos\beta+\mu\sin\beta-\dfrac{W}{R}}{\sin\beta-\mu\cos\beta}$$

whence, $\dfrac{R}{W}=\dfrac{\sin\beta+\mu\cos\beta}{2\mu}$.

Hence length of rod $=2\ PG=\dfrac{2a}{\cos\beta}\times\dfrac{R}{W}=\dfrac{a}{\mu}(\mu+\tan\beta)$.

Ex. 4. *A rectangular thin plate,* a *feet long and* h *feet high, is at rest on a rough horizontal plane. A string, attached to the top of the plate, is pulled horizontally with an increasing force. It is required to find whether the plate will tilt over before it slides or vice versa.*



Suppose the plate tilts (and does not slide) when the force is P.

(*i*) When it is on the point of tilting, it will do so by turning about B and therefore moments about B just balance. That is

$$Ph=W\frac{a}{2} \text{ or } P=\frac{Wa}{2h}.$$

(*ii*) Now, let us find what force will make the plate slide, supposing it does not tilt. When the plate is about to slide, there is limiting equilibrium and therefore,

$$W=R \text{ and } P=\mu R=\mu W.$$

Hence, *the plate will tilt first* if

$$\frac{Wa}{2h}<\mu W, \ \textit{i.e.}, \ \mu>\frac{a}{2h};$$

it will slide first if $\mu<\dfrac{a}{2h}$.

Ex. 5. *A uniform ladder is placed against a wall ; if the ground and the wall be* **equally rough** *and the co-efficient of friction be tan λ, show that the limiting inclination of the ladder to the vertical is 2λ.*

When the ladder is in this position, can it be ascended without its slipping ?

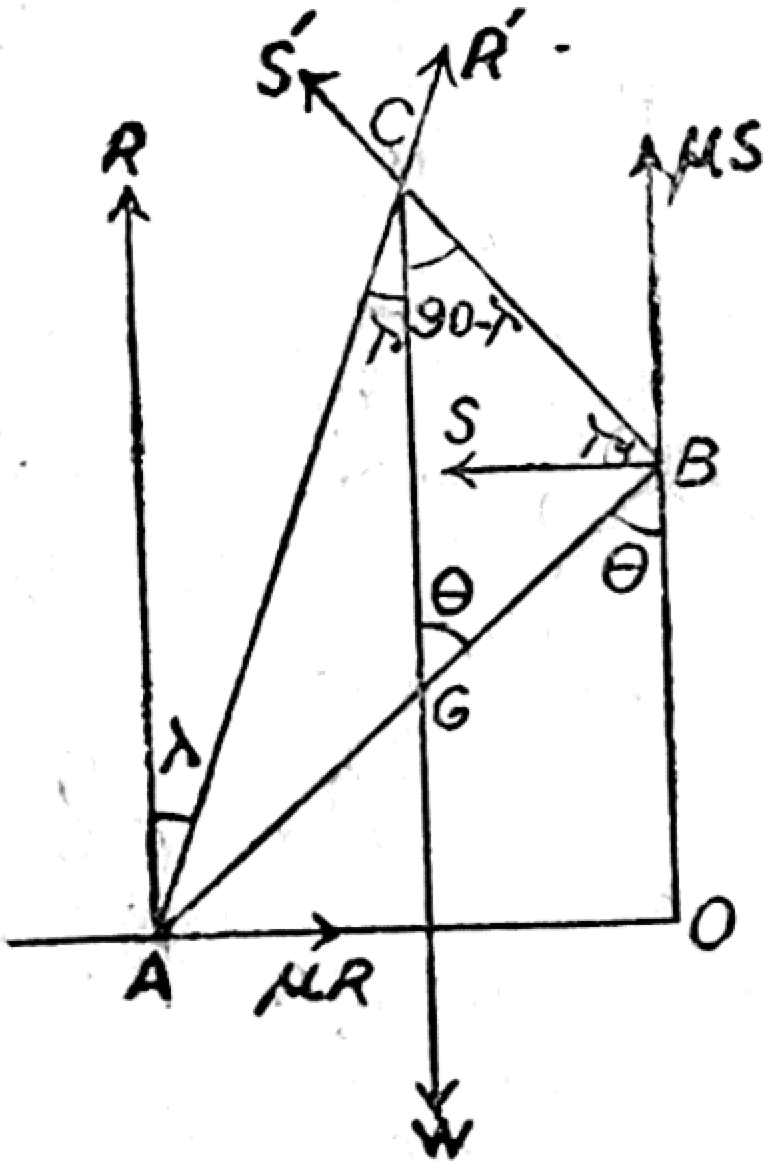
Let AB be the ladder and G, the mid-point of AB, its C. G. Five forces R, μR, W, S and μS act on the ladder and these are reduced to *three*, namely,

(*i*) R′, the resultant reaction at A,

(*ii*) S′, the resultant reaction at B,

and (*iii*) Weight of ladder W, acting vertically downwards through G.

These forces, therefore, must intersect in a point ; call it C. Since the ladder is on the point of slipping, R′ and S′ make an angle λ with the normals to the ground and the wall at A and B respectively as shown in the figure.



Applying formula

$$(m+n)\cot\theta = m\cot\alpha - n\cot\beta$$

to $\triangle$ ACB, we have

$$(AG+GB)\cot\theta = AG\cot\lambda - GB\cot(90^\circ-\lambda)$$

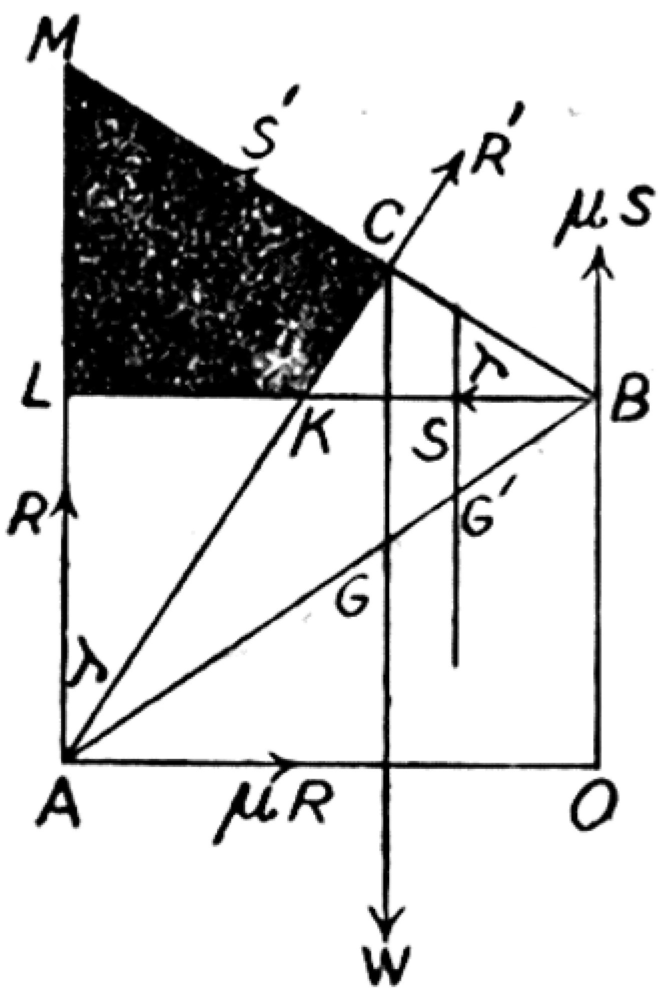
$$\therefore\ 2\cot\theta = \cot\lambda - \tan\lambda \qquad [\because\ AG=GB]$$

$$= \frac{1}{\tan\lambda} - \tan\lambda$$

$$= \frac{1-\tan^2\lambda}{\tan\lambda} = \frac{2}{\tan 2\lambda} = 2\cot 2\lambda.$$

Hence $\theta = 2\lambda$.

The resultant reactions at A and B can only meet within the shaded region CKLM, as their inclinations with the normals are not to exceed λ.



(*i*) Now, if the man, when ascending, is somewhere between A and G, the common C. G. of the man and the ladder will be shifted to a point below G. The vertical through the new C. G. will be to the left of the line CG, and can, therefore, meet the resultant reactions at A and B at some point in the region CKLM, thus satisfying the conditions of equilibrium of the ladder. *Equilibrium in this case, is non-limiting* as the resultant reactions make angles with the normals less than λ.

(*ii*) When the man is at G, the common centre of gravity will be at G and the vertical through G meets the resultant reactions at C. *Equilibrium in this case is limiting.*

(*iii*) When the man ascends to a point beyond G, G′ the common C. G. of the man and the ladder is between G and B and the resultant reactions at A, B cannot meet the vertical through G′ as this would require the ∠ of the resultant reaction at A to be greater than λ, which is impossible. The man can therefore ascend, only as far as G, the mid, pt. of AB.

Ex. 6. *A uniform ladder of length 2l rests against a rough vertical wall with its lower end on a rough horizontal plane, it being inclined to the wall at 45°. The co-efficient of friction between the ladder and the wall is $\frac{1}{3}$, and that between the ladder and the ground is $\frac{1}{2}$. If a man whose weight is one-half that of the ladder ascends the ladder, how high will he be when the ladder slips?*

Let G, the mid-point of AB the ladder, be its C. G., W its weight and C the position of the man when the ladder is on the point of slipping. Let R, S be the reactions at the ground and the wall, the frictions being μR and $\mu' S$.

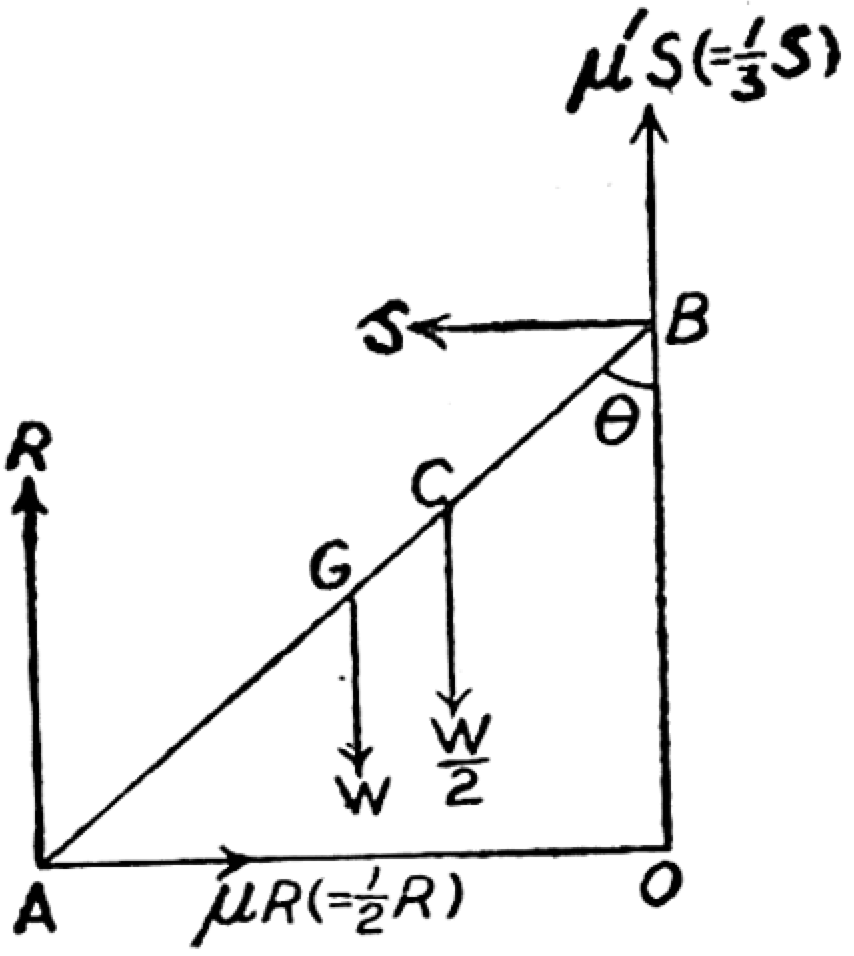
Here $\mu = \frac{1}{2}$, $\mu_1 = \frac{1}{3}$ and $\theta = 45°$.

AC$=x$ is to be found.

Resolving horizontally and vertically, we have

$$S = \frac{R}{2} \qquad \dots (i)$$

$$R + \tfrac{1}{3}S = \frac{3W}{2} \qquad \dots (ii)$$



Hence $R = \frac{2}{7}W$ and $S = \frac{9}{14}W$.

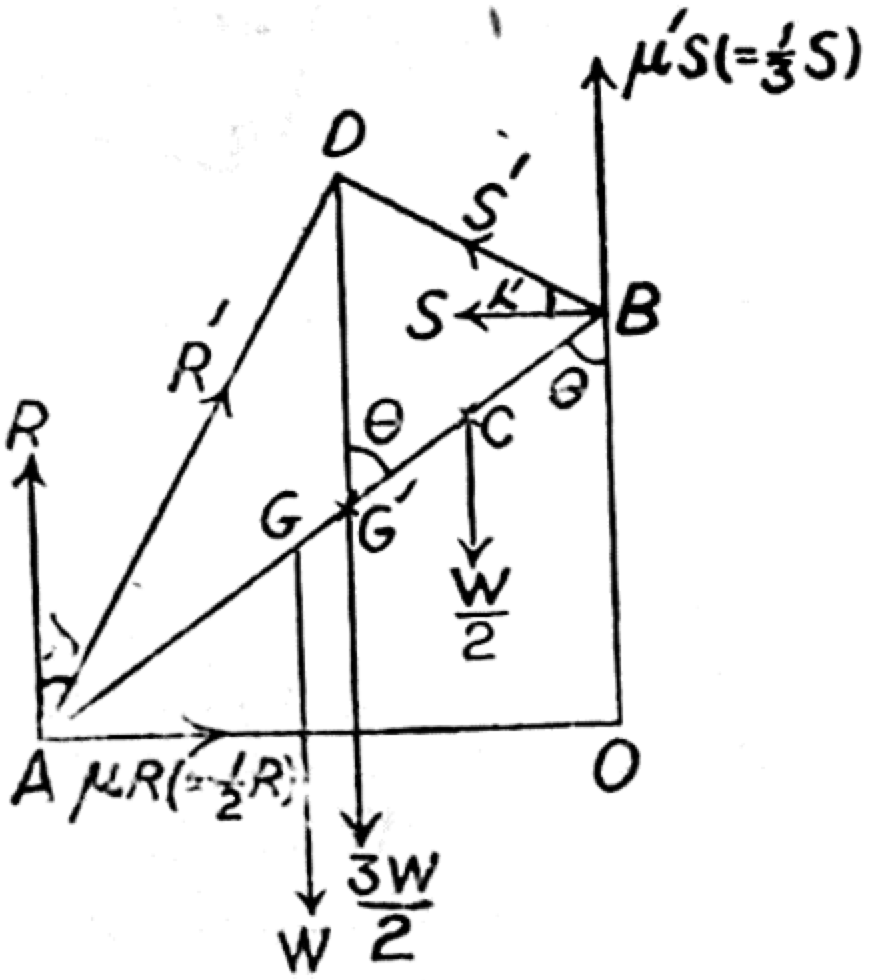
Taking moments about A, we have, when the ladder is in limiting equilibrium,

$$Wl \sin\theta + \tfrac{1}{2}W\, x \sin\theta = \tfrac{1}{3}S \times 2l \sin\theta + S \times 2l \cos\theta$$
$$= (\tfrac{2}{3}l \sin\theta + 2l\cos\theta)\, \tfrac{9}{14}W.$$

$$i.e., \ \tfrac{1}{2}x \sin\theta = \tfrac{3}{7}l \sin\theta + \tfrac{9}{7}l\cos\theta - l\sin\theta$$
$$\therefore \ x = \tfrac{6}{7}l + \tfrac{18}{7}l \cot\theta - 2l$$
$$= \tfrac{6}{7}l + \tfrac{18}{7}l - 2l \ (\because \cot\theta = \cot 45° = 1)$$
$$= \tfrac{10}{7}l = \tfrac{5}{7} \times 2l$$
$$= \tfrac{5}{7} \text{ of the length of the ladder.}$$

The man, therefore, cannot ascend the ladder beyond $\frac{5}{7}$th of its whole length from the foot.

Alternative Solution (*from geometrical considerations*).



Let the resultant reactions R′ and S′ at A and B respectively meet at D. Then, since the ladder is in equilibrium under 3 forces, the vertical through the common centre of gravity (G′) of the ladder and the man must pass through D.

We have, by applying the formula

$$(m+n)\cot\theta = m\cot\alpha - n\cot\beta$$

$$(AG' + G'B)\cot\theta = AG' \cot ADG' - G'B \cot BDG'.$$

But $\cot\theta = \cot 45° = 1$ and $\cot ADG' = 2$ $\cot BDG' = \frac{1}{3}$.

$$\therefore \ 2l = 2AG' - (2l - AG') \times \tfrac{1}{3}$$

$$\therefore \tfrac{8}{9}l = \tfrac{7}{9}\text{AG}'$$

$$\therefore \text{AG}' = \tfrac{8}{7}l\text{; hence BG}' = 2l - \text{AG}' = \tfrac{6}{7}.\ l.$$

If the ladder is on the point of slipping when the man has ascended the distance AC $= x$, then, since G′ is the common C. G. of the ladder and the man, we get taking moments about A,

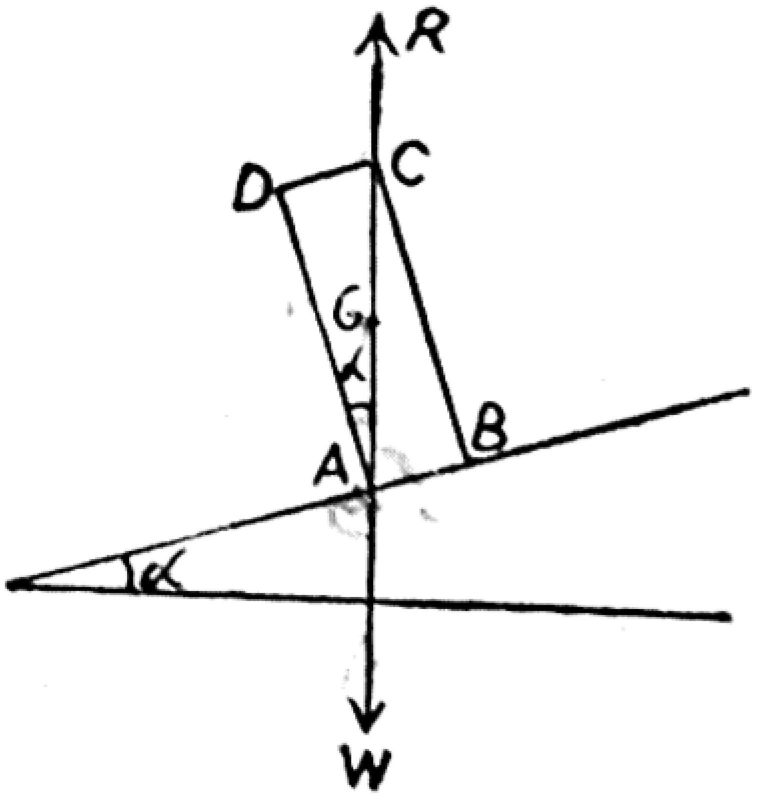
$$Wl + \frac{W}{2}\ x = \frac{3W}{2} \times \frac{8}{7}\ l$$

$$\therefore\ x = \frac{5}{7} \times 2l,$$

showing that the man cannot ascend the ladder beyond $\frac{5}{7}$th of its whole length from the foot.

Ex. 7. *A uniform cylinder of height h is placed with its plane circular base (of radius r) on a rough inclined plane and the inclination of the plane to the horizon is gradually increased. Determine how the equilibrium will be broken.* (*P. U. 1944*).

The equilibrium can be broken either by *tilting* or *by sliding*.



Let ABCD be the vertical section of the cylinder containing the line of action of the weight and α the inclination of the plane when the cylinder is on the point of tilting.

(*i*) The tilting evidently can take place only about the point A where R, the entire reaction of the plane on the cylinder, is concentrated, the base AB being loosely in contact with the inclined plane.

(*ii*) R and W, being the *only two forces* on the cylinder must be exactly equal and opposite when *tilting is about to take place*. Since R must pass through A, ∴ W acting, vertically downwards through G, must also pass through A. Observe that the line AG passes through C.

$$\text{Hence } \tan\alpha = \cot \text{GAB} = \frac{\text{AB}}{\text{CB}} = \frac{2r}{h} \qquad \dots (1)$$

(*iii*) Again, by Art. 135 inclination θ of the plane to the horizon when the cylinder is about to *slide* is given by

$$\tan\theta = \mu \qquad \dots (2)$$

(*iv*) If $\alpha < \theta$, toppling would take place before it slides.

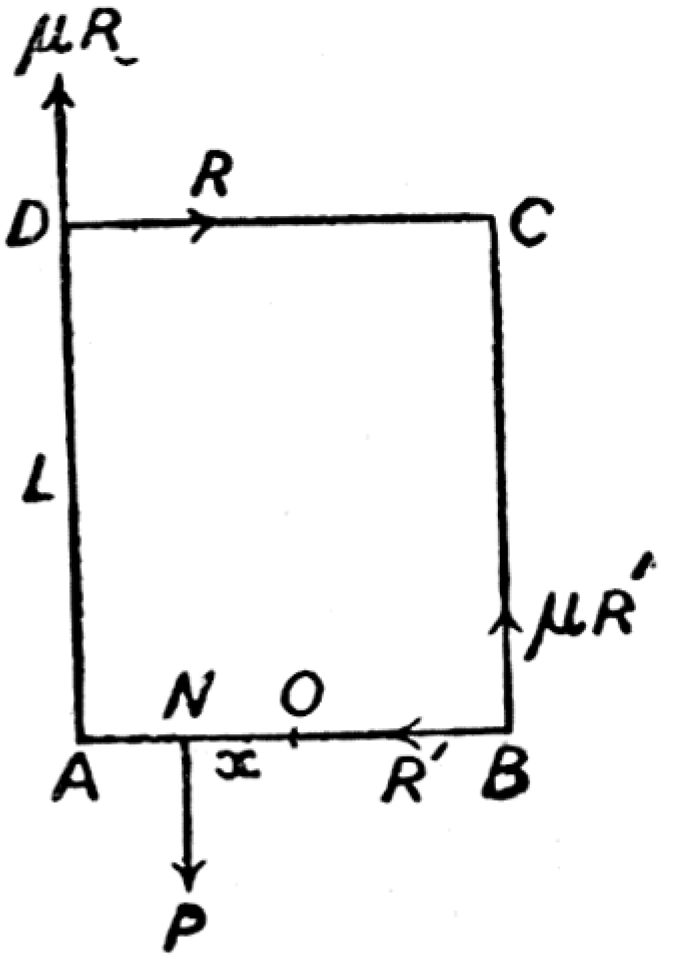
If $\theta < \alpha$, sliding will take place before it topples *i.e.*, the cylinder will topple before it slides if

$$\tan\alpha < \tan\theta,$$

i.e., if $\frac{2r}{h} < \mu$, and the reverse will be the case if $\frac{2r}{h} > \mu$.

Ex. 8. *A drawer is to be pulled out by a single force parallel to the drawer ; it is required to find how far from the middle of the end of the drawer, the force can be applied so as not to cause the drawer to jam.* (*P. U. 1938*)

Let l be the length of the drawer, AB the end of the drawer and O the mid point of AB.



Let N be the point where single force P must be applied so that jamming may just begin at the corners B and D. When this happens, resistance to motion is maximum and the values of friction at B and D are $\mu R'$ and μR, the thursts on the sides BC and AD being R' and R ; let ON$=x$ which is to be found.

The total resistances at B and D must meet on the line of action of P.

Since there is equilibrium, we have, by resolving parallel and perpendicular to AB

(*i*) $R = R'$ and

(*ii*) $P = \mu R + \mu R' = 2\mu R$ [from (*i*)

Taking moments about O.

(*iii*) $P.x + \frac{AB}{2}.\,\mu R' = R.\,l + \frac{AB}{2} \times \mu R$

i.e., $P.x = Rl$

$$= \frac{P}{2\mu} \times l \qquad \text{[from (ii)}$$

$$\therefore x = \frac{l}{2\mu}.$$

Note. If P is applied at a distance from O greater than x, the drawer will not move, however great P may be. If the drawer is very long or l great compared with AB, the point N may lie beyond A *i.e.*, the drawer will not jam at all.

Exercises XVII

1. A uniform ladder rests with one end on a horizontal floor and other against a vertical wall, the co-efficients of friction being respectively $\frac{3}{5}$ and $\frac{3}{5}$. Find the inclination of the ladder to the vertical when it is about to slip.

2. A uniform ladder rests in limiting equilibrium with one end on a rough floor, with co-efficient of friction μ, and the other against a smooth vertical wall. Find its inclination to the vertical.

3. A ladder rests in limiting equilibrium with one extremity against a vertical wall and the other on the ground, the inclination of the ladder to the vertical being 60°. Given that the ground and the wall are equally rough, find the co-efficient of friction.

4. A uniform ladder is placed against a wall. If the ground and the wall be equally rough, angle of friction being λ, (and if the ladder is about to slip), find its inclination to the vertical.

5. A uniform ladder rests in limiting equilibrium against a smooth vertical wall, the ground being rough. Find how high a man can now ascend the ladder.

6. A ladder whose centre of gravity divides it into two portions of length a and b, rests in limiting equilibrium, with one end on a rough horizontal floor and the other against a rough vertical wall. If the co-efficients of friction at the

floor and the wall be respectively μ and μ', find the inclination of the ladder to the floor.

7. Prove that the horizontal force which will just sustain a heavy particle on a rough inclined plane will sustain the particle on the same plane supposed smooth, if the inclination is diminished by the angle of friction.

8. A body of weight W can be just sustained on a rough inclined plane by a horizontal force P; it can also be just sustained on the same plane by a force Q up the plane. Find the angle of friction.

9. A force P acting up a rough plane can just sustain on it a body of weight W and a force Q acting up the plane can just drag the body up the plane. Find the angle of friction.

10. Two equal uniform rods AC, CB, each of weight W, lean against each other and rest in a vertical plane with ends A and B on a rough horizontal plane. The angle ACB is 2α and the co-efficient of frictiou μ. Find what weight placed at C would cause them to slip.

11. In question 10, if the weight be placed at the middle point of one rod and be heavy enough to cause slipping, show that the other rod will be the one that will slide first.

12. A bead runs on a rough circular wire whose plane is vertical; if a is the radius of the circle, and λ the angle of friction, prove that the bead can be in equilibrium at any point of certain arcs whose lengths are $2a\lambda$ each.

13. A heavy particle is placed on a rough inclined plane, of inclination $\cos^{-1}\frac{4}{5}$, and is connected by a string passing over a smooth pulley with a particle of equal weight, which hangs freely. Assuming that the particle on the plane is about to move up the plane, find the inclination of the string to the plane, the co-efficient of friction being $\frac{1}{3}$.

14. A particle of weight W is just sustained on an inclined plane of inclination a by a force P. Find the angle that P makes with the inclined plane when the normal reaction on the plane is W, co-efficient of friction being μ.

15. A heavy uniform ladder rests with one end on the rough ground and the other end against an equally rough vertical wall; find the least co-efficient of friction that will allow the ladder to rest in all positions.

16. A rectangular lamina PQRS, whose height is double its base, stands with its base PS on a rough floor, co-efficient of friction being $\frac{1}{6}$. If it is pulled by an increasing horizontal force at R, find *whether it will first slide or tilt.*

17. A cylinder rests with its circular base on a rough inclined plane, the co-efficient of friction being $\frac{1}{2}$. Find the inclination of the plane and the relation between the height and the radius of the base of the cylinder, *so that it may be on the point of sliding as well as of toppling over.*

18. A rectangular block is placed, with one of its edges horizontal, on a rough plane, the inclination of which to the horizon is gradually increased; *find whether the equilibrium of the block will be broken by a motion of sliding or of toppling.*

19. A cone of semi-vertical angle α rests on a rough table and a string fastened to the vertex of the cone passes over a smooth pulley and supports a weight. Show that if the string is horizontal and the weight is gradually increased, the cone *will turn over or slide* according as $\mu >$ or $<$ $\tan \alpha$.

20. AC and CB are two uniform rods of the same length but of different weights P and Q respectively freely jointed at C: they are in the same vertical plane and AC is movable round a smooth horizontal axis through A and the end B slides along a rough horizontal plane in which A also lies. Find the limiting position of equilibrium.

21. A particle is tied to one end of a light inelastic string, the other end of which is fixed to a point in a rough plane inclined to the horizon at angle α, which is greater than the angle of friction λ. Prove that the particle cannot remain at rest if the angle between the string and the line of greatest slope is greater than $\sin^{-1}(\tan \lambda \cot \alpha)$.

22. A uniform heavy bar, 12 inches long, one end of which is smooth and the other is rough, rests in limiting equilibrium within a circular hoop of radius 10 inches in a vertical plane, the rough end of the bar being at the lowest point of the hoop. Find the co-efficient of friction.

23. A heavy uniform rod is in limiting equilibrium while resting on two rough pegs P and Q, PQ being inclined at an angle α to the horizon. Show that the centre of gravity of the rod divides PQ in the ratio $\mu - \tan \alpha : \tan \alpha - \mu'$, given

that co-efficients of friction at P and Q are respectively μ and μ' and $\mu > \tan\alpha > \mu'$.

24. A rod of length a and weight W is supported in a horizontal position by two strings of length a attached to the ends of the rod to two rings of weight $\frac{1}{4}$ W which are free to move along another fixed rough horizontal rod. If μ be the co-efficient of friction between the rings and the rod, show that the greatest distance apart of the rings consistent with equilibrium is

$$a\left(1+\frac{6\mu}{\sqrt{9\mu^2+4}}\right).$$

25. The two ends of a heavy uniform bar are in contact with two planes inclined at an angle of 60° and 30° to the horizon respectively. Prove that the bar can just rest horizontally, if either plane is smooth and the co-efficient of friction of the other is $\frac{1}{\sqrt{3}}$.

26. A uniform rod rests in limiting equilibrium within a rough vertical circle; if the rod subtends an angle 2α at the centre of the circle and if λ be the angle of friction show that the angle of inclination of the rod to the horizon is

$$\tan^{-1}\frac{\sin 2\lambda}{\cos 2\lambda+\cos 2\alpha}.$$

[Hints. $\angle AOG=\angle GOB=\alpha$. [$\because$ OA=OB (radii)]

Let the *resultant* reactions R′ and S′ at A and B meet at C.

The rod AB is in equilibrium under the action of the three forces R′, S′ and W which must intersect at C.

$\angle CAG=\angle OAG-\lambda=90°-\alpha-\lambda$ $\quad\because \angle OAG=\angle OBG$
$=90°-\alpha$

$\angle CBG=\angle OBG+\lambda=90°-\alpha+\lambda$

$\angle CGB=90°-\theta$.

$\therefore$ from $\triangle$ CAB, since AG : GB=1 : 1, we have

$$(1+1)\cot(90°-\theta)=1.\cot(90°-\alpha-\lambda)-1.\cot(90°-\alpha+\lambda)$$

$$2\tan\theta=\tan(\alpha+\lambda)-\tan(\alpha-\lambda)$$

$$=\frac{\sin 2\lambda}{\cos(\alpha+\lambda).\cos(\alpha-\lambda)}$$

$$\therefore\ \tan\theta = \frac{\sin 2\lambda}{\cos 2\alpha + \cos 2\lambda},$$

$$= \frac{\sin 2\lambda}{\frac{1}{2}\cos(\alpha+\lambda+\alpha-\lambda)+\cos(\alpha+\lambda-\alpha+\lambda)}$$

$$= \frac{2\sin 2\lambda}{\cos 2\alpha + \cos 2\lambda}.\ ]$$

27. Two bodies of weights P and Q are placed on an inclined plane and are connected by a light string which lies along a line of greatest slope of the plane ; if the co-efficients of friction be respectively μ and μ', $(\mu > \mu')$ find the inclination of the plane when both bodies are on the point of motion, it being assumed that the body P is lower than the other.

28. A heavy body is to be dragged up a rough inclined plane ; find the direction of the least force that will suffice, *i.e.*, the best angle of traction up a rough inclined plane. [See Ex. 7 worked out].

29. A weightless rod of length l, is supported horizontally between two rough inclined planes at right angles to each other, the angle of friction λ being less than the inclination of either plane. Show that the length of that position of the rod on which a weight may be placed without producing motion is $l \sin 2\alpha \sin 2\lambda$, α being the inclination of either plane to the horizon.

30. A heavy uniform bar of length $2l$ lies over a rough peg with one end resting against a rough vertical wall ; if a be the distance of the peg from the wall and λ be the angle of friction at the peg as well as at the wall, show that when the point of contact of the bar with the wall is above the peg, then the bar is about to slide downwards when $l \sin^2\theta = a \cos^2\lambda$, where θ is the inclination of the bar to the wall.

If the point of contact of the bar and the wall be below the peg, prove that the bar is about to slide downwards when $l \sin^2\theta \sin(\theta+2\lambda)=a\cos^2\lambda$, and about to slide upwards when

$$l \sin^2\theta \sin(\theta-2\lambda)=a\cos^2\lambda.$$

31. A uniform bar slides with its ends on two fixed equally rough inclined planes, one being vertical and the other inclined at an angle α to the horizon. Show that the inclina-

tion θ to the vertical of the bar, when it is on the point of sliding is given by

$$\tan\theta = \frac{2(\tan\alpha \pm \mu)}{1 \mp 2\mu \tan\alpha - \mu^2},$$

it being assumed that the vertical plane through the bar cuts the planes along lines of greatest slope.

32. Two uniform rods AB, BC, each of length $2a$, and of weights W_1 and W_2 respectively ($W_1 < W_2$), are freely jointed together at B. The system stands in a vertical plane, with A and C on a rough horizontal table, the rods each making an angle θ with the vertical. The co-efficient of friction both at A and C is equal to μ. Show that equilibrium is possible only if

$$(3W_1 + W_2)\mu - (W_1 + W_2)\tan\theta \geqslant 0.$$

This condition being satisfied, a weight W is suspended from B. Show that, if $\mu \geqslant \tan\theta$, equilibrium is still possible whatever the value of W, but that if $\tan\theta > \mu$, equilibrium is no longer possible if

$$W > \frac{(3W_1 + W_2)\mu - (W_1 + W_2)\tan\theta}{2(\tan\theta - \mu)}.$$

33. A light shallow rectangular drawer is of width a, while its length from front to back is l. It can be pulled out by two knobs attached to the front and at equal distances f from the sides. If a tension parallel to the sides is applied to only one knob, prove that the drawer will jam if the co-efficient of friction between the sides and thec asing exceed

$$\frac{1}{a - 2f}.$$

34. A uniform rod is in limiting equilibrium, one end resting on a rough horizontal plane and the other on an equally rough plane inclined at an angle α to the horizon. If λ be the angle of friction, and the rod be in a vertical plane, show that the inclination θ, of the rod to the horizon is given by

$$\tan\theta = \frac{\sin(\alpha - 2\lambda)}{\cos(\alpha - 2\lambda) - \cos\alpha}.$$

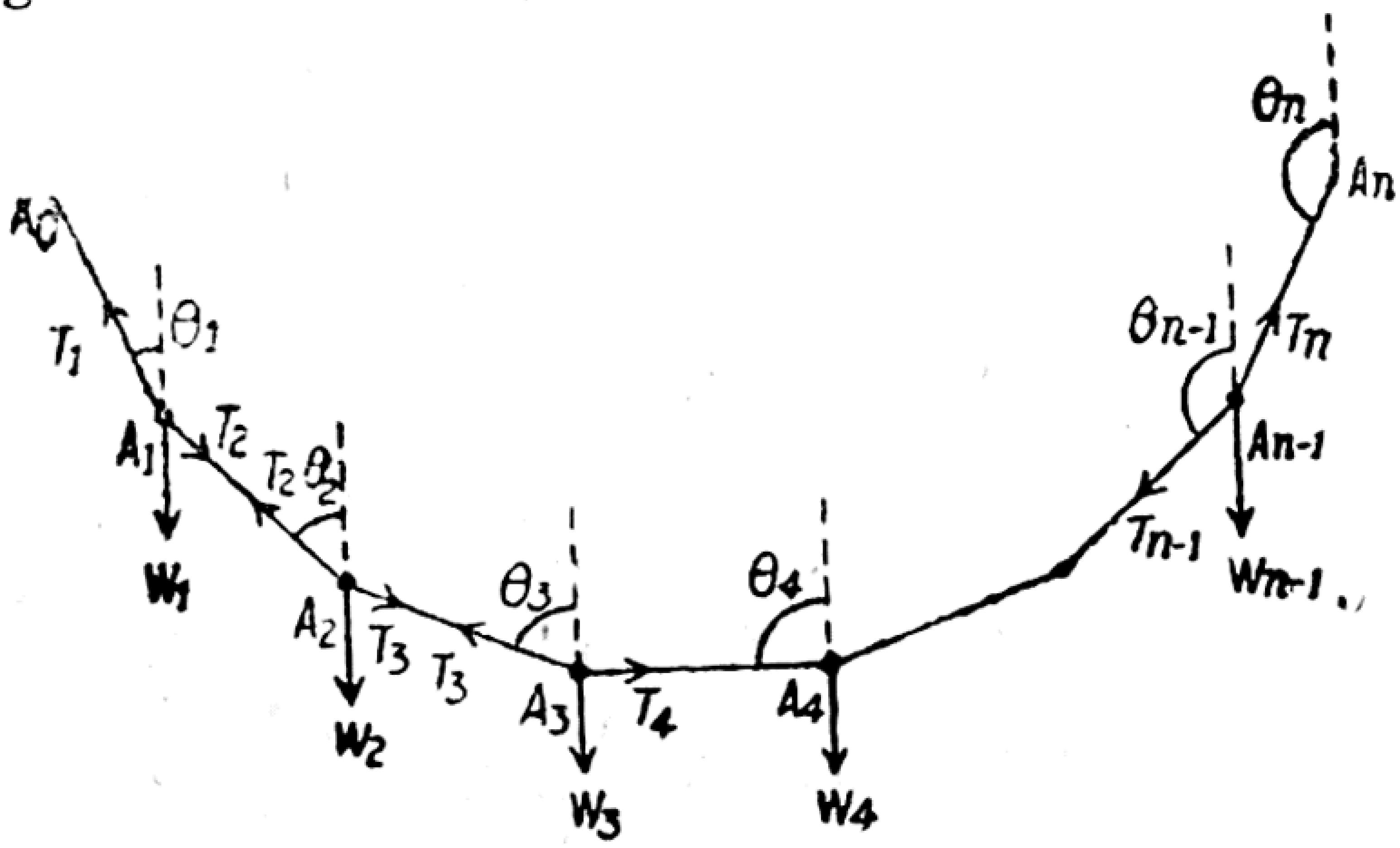
35. A uniform rod of weight W is placed with its lower end on a rough horizontal floor and its upper end against an *equally rough* vertical wall, the rod makes an angle α with the wall and is just prevented from slipping down by a horizontal force P applied at its middle point. Prove that $P = W \tan(\alpha - 2\lambda)$ where λ is the angle of friction and $\lambda < \frac{1}{2}\alpha$ (*P. U. 1943*).

36. A uniform ladder of length l and weight W rests with its foot on rough ground and its supper end against a *smooth* wall, the inclination of the ladder with the vertical being α. A force P is applied horizontally to the ladder at a point distant C from the foot so as to draw the foot to the wall. If μ be the co-efficient of friction at the foot, prove that P must exceed $\frac{Wl}{l-C}(\frac{1}{2}\tan\alpha + \mu)$. (*P. U. 1942*).

CHAPTER XII

STRINGS

141. Funicular Polygon. If a weightless string be fastened at its extremities to fixed points and a number of weights be attached to the string at different points, the polygonal figure assumed by the string is called *Funicular polygon*. We now proceed to find tensions in different parts of the string, and also their inclinations, in such a case. Let the weights W_1, W_2, W_3 ....., W_{n-1} be attached at the points A_1, A_2, A_3......A_{n-1} ; A_0 and A_n being the extremities of the string.



Let the parts A_0A_1, A_1A_2, A_2A_3.....$A_{n-1}A_n$ be of lengths a_1, a_2, a_3......a_n, respectively, and let them make angles θ_1, θ_2, θ_3......θ_n with the vertical, in the same direction.

Now at each of the points A_1, A_2......A_{n-1}, we have got *three* forces in equilibrium. We might proceed to construct a series of triangles, each of which will correspond to the forces at a point and obtain therefrom the tensions and the inclinations ; or, we might solve the problem *analytically* by resolving the forces horizontally and vertically.

Using the analytical method, we have the following $2n-2$ equations :

$$T_1 \sin \theta_1 = T_2 \sin\theta_2 = T_3 \sin \theta_3 = \ldots\ldots$$
$$= T_n \sin \theta_n \ldots\ldots (A)$$

$$\left.\begin{aligned} \text{and } T_1 \cos\theta_1 - T_2 \cos\theta_2 &= W_1 \\ T_2 \cos\theta_2 - T_3 \cos\theta_3 &= W_2 \\ T_3 \cos\theta_3 - T_4 \cos\theta_4 &= W_3 \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ T_{n-1} \cos\theta_{n-1} - T_n \cos\theta_n &= W_{n-1} \end{aligned}\right\} \quad \ldots\ldots(B).$$

Equations (A) *show that horizontal components are the same throughout, and equations (B) that the consecutive vertical components differ by the weights attached respectively*. But these $2n-2$ equations are not sufficient to give the n tensions and the n inclinations. We require 2 more equations which may be written by considering the horizontal and the vertical distances between A_0 and A_n.

Let these distances be h and k, respectively.

Then $a_1 \sin\theta_1 + a_2 \sin\theta_2 + a_3 \sin\theta_3 + \ldots + a_n \sin\theta_n = h$

and $a_1 \cos\theta_1 + a_2 \cos\theta_2 + \ldots\ldots + a_n \cos\theta_n = k$,

where k is the distance by which A_0 is vertically *above* A_n.

If the constant horizontal components be taken $= \lambda$, we have from (B),

$$\cot\theta_1 - \cot\theta_2 = \frac{W_1}{\lambda},$$

$$\cot\theta_2 - \cot\theta_3 = \frac{W_2}{\lambda},$$

$$\cot\theta_3 - \cot\theta_4 = \frac{W_3}{\lambda},$$

and so on

If $W_1 = W_2 = W_3 \ldots\ldots = W_n$, the right hand sides of the above equations being equal, $\cot\theta_1$, $\cot\theta_2$, $\cot\theta_3$ form a series in A. P., or, which is the same thing, *the tangents of the angles which the different parts of the string make with the horizontal, are in A. P.*

The above theory of funicular polygon is applicable to the case of a suspension bridge, where tensions in a number of vertical strings which are attached to the horizontal span of the bridge take the place of the weights. In this case, the extreme portions of the wires pass over high piers at each end, A_0 and A_n and are secured in the ground beyond. A_0, A_n being in the same horizontal line, $k=0$ and h measures obviously the span of the bridge.

142. A string or chain is said to be **flexible**— rather perfectly flexible—when it does not offer any resistance whatever to bending. Thus any force, however small, which is applied otherwise than along the direction of such a string will change its form. And the force exerted by one particle of the string on the next is in the direction joining the two particles. It is evident that like ideal rigidity, ideal flexibility is only partially realised in practice.

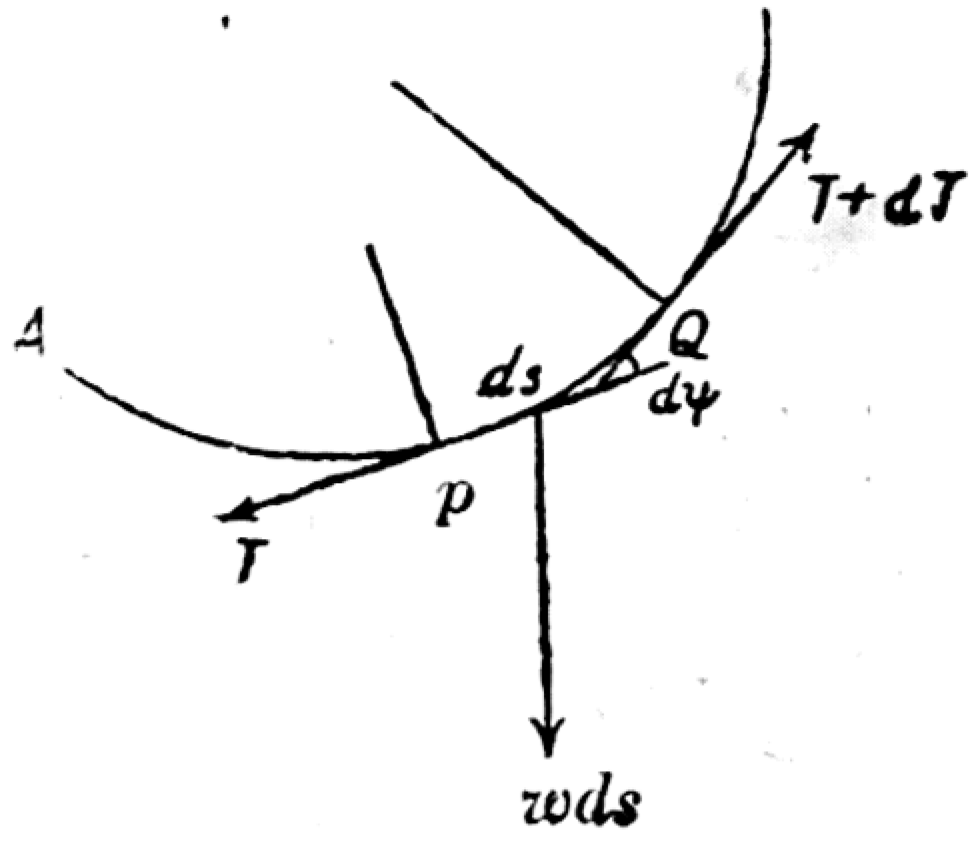
Since the tension of a string acts at every point of the string and tends to stretch it in the direction of its length, it may or may not yield to this tendency to stretch.

A string which does not stretch under tension is called *inextensible* ; a string which stretches under tension is called *extensible.*

We may also extend the theorem of Art. 141 in considering the *curve* which will be formed by a heavy string of perfect flexibility, which hangs freely between two points of suspension. When the string has a uniform density, the curve thus formed is known as *Catenary* or *Chainette.*

143. Problems *: A flexible uniform string is stretched over a given plane curve ; find the tension at any point and the pressure on the curve.*

Let us consider the equilibrium of the element PQ$=ds$ of a flexible uniform string of weight w per unit length stretched on the curve APQ in a vertical plane, S being measured from some fixed point A on the string. Let the x-axis be horizontal. The element PQ is acted on by the following forces :



(1) its own weight wds vertically downwards.

(2) the tensions T at P and T$+d$T at Q,

(3) the reaction or resistance Rds of the curve on PQ which will be ultimately along the normal at P,

(4) the friction $F\,ds$ which will be ultimately along the tangent at P and in the direction opposite to that in which the element is about to move.

Resolve the forces along the tangent and normal at P and we have

$$(T+dT)\cos d\psi - T - F\,ds - w\,ds\sin\psi = 0$$
$$(T+dT)\sin d\psi - R\,ds - w\,ds\cos\psi = 0$$

$\therefore$ ultimately

$$\frac{dT}{ds} = F + w\sin\psi, \qquad \frac{T}{\rho} = R + w\cos\psi,$$

where ρ is the radius of curvature at P

Special Cases.

Case (i) Assume that the curve is smooth and the string is light.

The equations of equilibrium take the form

$$dT = 0, \text{ and } R = T/\rho.$$

Thus the tension is constant and the pressure at any point varies as the curvature there.

Since T is constant the string will not be in equilibrium unless the forces pulling at its two ends are equal.

Case (ii) Suppose that the curve is smooth and the string is heavy.

Let w be the weight per unit length.

The equations of equilibrium are

$$dT = w\,ds\sin\psi = w\,dy \;\ldots\ldots\ldots\ldots(1)$$

$$\text{and} \quad R = \frac{T}{\rho} - w\cos\psi \;\ldots\ldots(2)$$

From (1), $T = wy + C$.

$\therefore$ the difference in tensions at two points varies as the difference of their levels.

Case (iii) Let the curve be rough and the string light. The equations of limiting equilibrium are

$$\frac{dT}{ds} = \mu R, \text{ and } T\frac{d\psi}{ds} = R,$$

where μ is the co-efficient of friction.

$$\text{Thus} \quad \frac{1}{T}\frac{dT}{d\psi} = \mu.$$

Integrating, $\log T = \mu\psi + \text{a constant}$

$$\therefore \quad T = Ce^{\mu\psi}.$$

Case (iv) Assume the curve to be rough and the string heavy.

Let w be the weight per unit length.

The equations of limiting equilibrium are

$$\frac{dT}{ds} = \mu\, R + w \sin\psi$$

and $$\frac{T}{\rho} = R + w \cos\psi.$$

where μ is the co-efficient of friction.

$$\therefore \quad \frac{dT}{d\psi} - \mu\, T = w\rho\,(\sin\psi - \mu\cos\psi).$$

Thus for finding T, we *have a differential equation* which can be solved in any particular case.

144 Elastic Strings : Hooke's Law. All strings and rods that we have considered hitherto in the equilibrium of a body have been supposed to be in-extensible, however great the forces acting on the body. But, as a matter of fact, they all yield to the action of forces and are strained, more or less, from their natural shape. Also, if the force causing a strain does not exceed a certain limit, the string or the rod has a tendency to recover its originnl form, when the force is removed. Such extensible strings or rods are also called *elastic.*

The string or the rod increases in length as the force causing tension along it increases. The relation between the force of tension and the extension of the string or the rod is stated by a law due to an English Mathematician, Robert Hooke (1635-1703). The law, which is therefore known as **Hooke's Law**, enunciates simply that *Tension varies as Extension.* If l be the *natural length* and l' the *stretched length*, the extension is $l'-l$. Hooke's Law may then be represented symbolically in the form $T \propto l'-l$, (where T is the tension)

$$\text{or } T = k(l^1 - l) \ldots\ldots\ldots\ldots (1)$$

$T \propto E$

This constant of variation, k, is usually taken as $\frac{E}{l}$, and thus the above takes the form

$$T = E.\frac{l'-l}{l} \text{.................}(2)$$

This constant E depends only on the thickness and material of the string and is known as *Young's Modulus of the string* or *Modulus of Elasticity of the string.*

It is not difficult to show that E is not the same for all bodies of the same material. The modulus is proportional to the area of cross-sections.

The rod which will obey Hooke's Law must be thin and straight, and the extensions very small.

It is obvious that E, in (2) is itself a force. For let $l' = 2l$, then (2) gives that $T=E$, *i. e., the modulus is a tension which extends a string to double its natural length.* In other words, the force required to stretch a string to double its natural length is called the modulus of elasticity of the string, provided that the formula remains true for so large an extensibility.

For a rod, E will be very large and will, obviously, depend upon the area of the cross-section. For a steel bar of 1 sq. in. section, E = 13200 tons.

If E and T be known, the extended length is from (2), given by $l' = l\,(1+T/E)$.

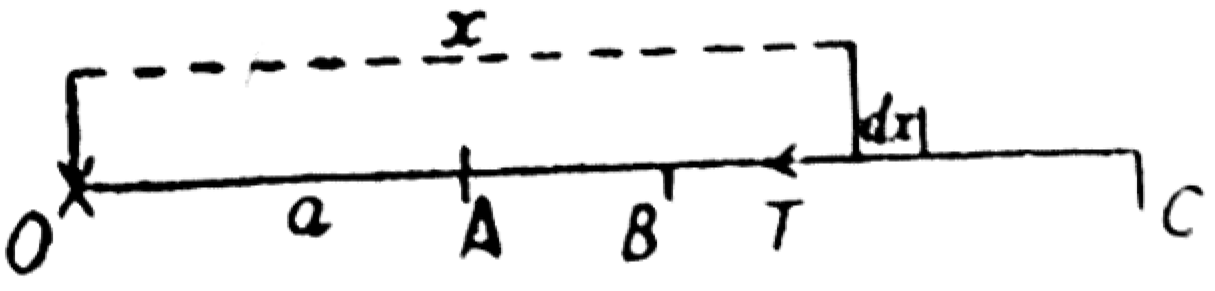
145. Work done in stretching an elastic string.

Let the *natural* (unstretched) *length* of the string be a, and λ be the modulus of elasticity. When the string is stretched to a length x, its tension is

$$T = \lambda\frac{x-a}{a}\text{, where } \lambda \text{ is the modulus of elasticity.}$$

In stretching it from x to $x+dx$, *work is done against the tension* and is equal to

$$Tdx = \frac{\lambda}{a}\,(x-a)\,dx.$$



The total work done in stretching the string from length b (greater than the natural length) to length c,

$$= \int_b^c \frac{\lambda}{a}\,(x-a)\,dx$$

c − a = [illegible]
c − a [illegible]

$$= \frac{\lambda}{a}\int_b^c (x-a)dx$$

$$= \frac{\lambda}{a}\left|\frac{(x-a)^2}{2}\right|_b^c = \frac{\lambda}{a}\left\{\frac{(c-a)^2}{2}-\frac{(b-a)^3}{2}\right\}$$

$$= \frac{\lambda}{2a}\left\{(c+b-2a)(c-b)\right\}$$

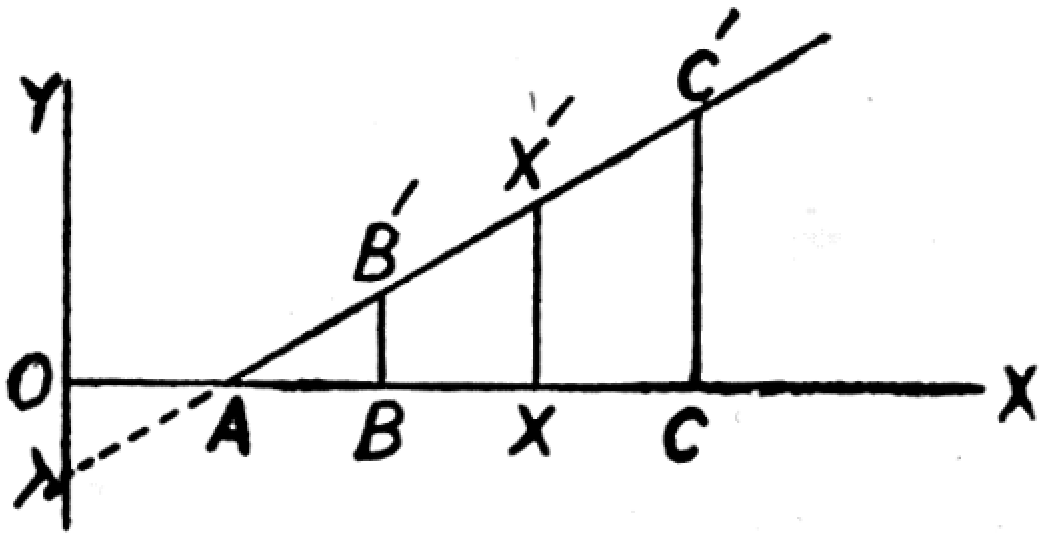
$$= \frac{(c-b)}{2}\left\{\frac{\lambda}{a}(c-a)+\frac{\lambda}{a}(b-a)\right\}$$

$$= (c-b)\left\{\frac{T_c+T_b}{2}\right\}$$

=*extension produced multiplied by the mean of the initial and final tensions.*

145. We can employ the **Graphical method** for finding *the work done in stretching an elastic string* from length OB to length OC.

Let OA=a=*natural length of the string,* OB=b ; OC=c ; and OX=x.



When the string is stretched from its natural length OA to OX, the tension of the string is given by

$$T=\lambda\,\frac{x-a}{a}\text{ (Hooke's law),}$$

showing that T is function of x.

Representing T by y, Hooke's law gives

$$y=\lambda\,\frac{x-a}{a}=\frac{\lambda}{a}\,x-\lambda,$$

which is a straight line (being an equation of the first degree in x and y) showing that the graph representing the relation between tension T and extension x is *a straight line.* The slope of the straight-line graph (m of the line) is $\frac{\lambda}{a}$ and its intercept on the y-axis is $-\lambda$. BB′, XX′, CC′ are the tensions when the extended lengths, are OB, OX and OC respectively.

Work done in stretching the string from OB to OC $=$ BB′C′ CB (Art. 90).

$=$area of trapezium BB′C′C

$$=\text{BC} \times \left(\frac{\text{BB}'-\text{CC}'}{2}\right)$$

$=$*extension produced multiplied by the mean of the initial and the final tensions*;
since, BC$=$extension produced, BB′$=$initial tension, BC′$=$ final tension.

147. Ex. 1. *The upper end of a light string hanging vertically is held fast; and a weight of 2 lbs. attached to the lower end is observed to extend it by one-third its original length. Find the extension of the string when a weight of 6 lbs. is attached to the lower end.*

Here $\frac{(l'-l)}{l}=\frac{1}{3}$ $\therefore$ 2 lbs. wt $=\frac{1}{3}$E.

Hence the modulus of elasticity$=$6 lbs. wt.

$\therefore$ $6=6.\frac{l'-l}{l}$ $\therefore$ $l'=2l$ and the extension is equal to the original length.

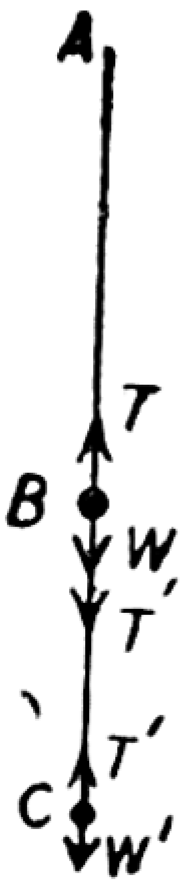
Ex. 2. *A light string ABC hanging vertically has its upper end A held fast: if weights W and W′ attached to points B and C (C being lowest), extend the portion AB and BC, respectively, by equal and one-half of their original lengths, find the relation between W and W′.*

Let T and T′ be the tensions in AB and BC.
Then T′$=$W′ and T$=$T′$+$W$=$W$+$W′

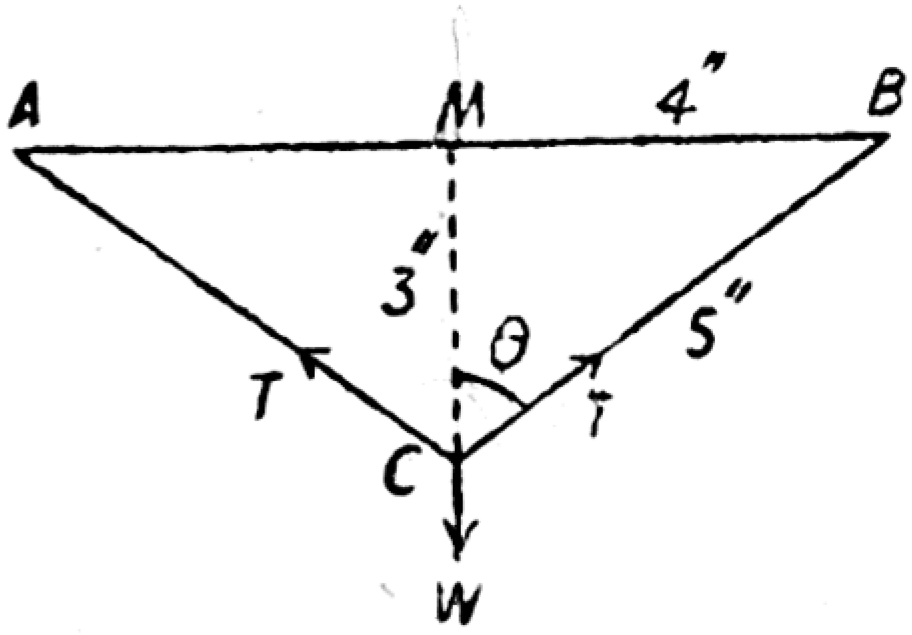
$\therefore$ T$=$E. $\frac{l'-l}{l}=$E, E$=$W$+$W′.

Again, $\because T' = E.\frac{1}{2} = W'$, $E = 2W'$.

$\therefore W + W' = 2W'$, *i.e.*, W and W′ are equal and the modulus of elasticity is double each weight.



Ex. 3. *A weightless string has its two ends attached to two points, 8 inches apart, in the same horizontal line, and the string is straight. If a ring, which is made to slide on the string brings down its middle point by 3 inches, find the modulus of elasticity and the tension in the string. Find also the work done.*



Let W lb. be the weight of the ring, T the tension of the string and θ the angle MCB.

Then $2T\cos\theta = W$ and

$$\cos\theta = \frac{3}{5},$$

$$\therefore T = \frac{5}{6} W \text{ lbs.}$$

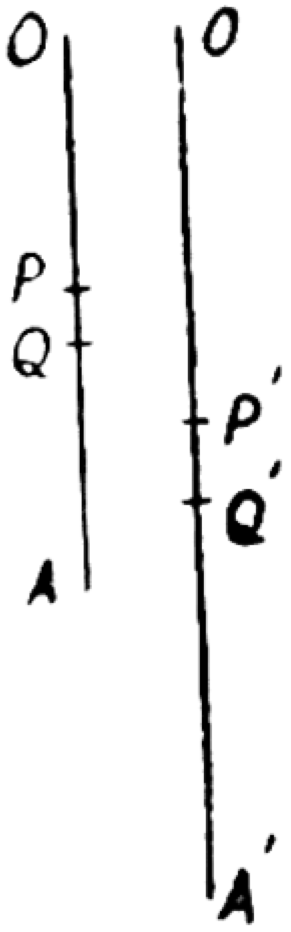
$$\text{But } T = E.\frac{10-8}{8} = \frac{E}{4}.$$

$$\therefore \frac{5}{6}W = \frac{E}{4}, \text{ i.e., } E = \frac{10}{3} W \text{ lbs.}$$

As the ring is going vertically downwards, the direction of the tension is continually changing. So here we have both *magnitude* and *direction* of the tension variable. Hence Art. 148 is not applicable. The work done is found simply by considering the vertical distance through which the ring goes down.

$$\text{Hence work done} = W \times \frac{3}{12} = \frac{W}{4} \text{ ft. lbs.}$$

Ex. 4. *An elastic string of weight W has a weight W attached to one end and it is hung vertically from the other end; determine the extension of the string.*



Let $OA = l$ be the unstretched length of the string; OA' the stretched length. Let $OP = x$, $PQ = \delta x$.

Suppose OP stretched into OP' and PQ into $P'Q'$; let $OP' = x'$, $P'Q' = \delta x'$.

Let T and $T + \delta T$ be the tensions at P', Q'. Then the element $P'Q'$ is acted upon by three forces: T, $T + \delta T$ at its ends and its own weight; its weight is equal to that of PQ which is $\frac{\delta x}{l} w$.

$$\therefore T + \delta T - T + \frac{\delta x}{l} w = 0.$$

$$\text{or } \frac{dT}{dx} = -\frac{w}{l} \text{ ultimately} \qquad \text{......(1)}$$

$$\therefore T = -\frac{wx}{l} + C$$

When $x = l$, $T = W$; $\therefore C = W + w$,

$$\text{and } \therefore T = W + w\left(1 - \frac{x}{l}\right) \qquad \text{......(2)}$$

This equation may also be obtained by observing that the tension at any point must be equal to the weight below that point.

Also by Hooke's Law,

$$T = E\,\frac{\delta x' - \delta x}{\delta x} \qquad \text{......(3)}$$

$$\therefore \frac{dx'}{dx} = 1 + \frac{T}{E} = 1 + \frac{W}{E} + \frac{w}{E}\left(1 - \frac{x}{l}\right).$$

Integrating, we get

$$x' = x\left(1 + \frac{W + w}{E} - \frac{wx^2}{2El}\right) + C.$$

When $x = 0$, $x' = 0$, and $\therefore C = 0$.

If l' be the stretched length, then

$$l' = l\left(1 + \frac{W + w}{E}\right) - \frac{wl}{2E}$$

$$= l\left(1 + \frac{W + \frac{1}{2}w}{E}\right).$$

Exercise XVIII

1. A string 10″ long extends 14″ under the action of a force equal to the weight of 4 lbs. If the force be increased to 6 lbs. wt., find the increase in the extension and the work done on account of this increase.

2. The upper end of a string hanging vertically is fixed and equal weights are attached to the middle point and to the lower end. If the modulus of elasticity be also equivalent to the weight at one of the points, show that while the lower portion of the string is doubled, the upper is trebled.

3. An elastic string 5″ long can be extended by 1″ under the action of a force of 2 lbs. wt. If the length of the string were increased by 3″, find the total work done from the original position of the string.

4. A straight wire ABC is held fast at A and allowed to hang vertically ; weights of 5 and 7 lbs., respectively, are attached to B and C. If AB and BC be 4″ and 2″, respectively, find the ratio of the work done in stretching AB to that done in stretching BC.

5. An elastic string when tied to two points in the same horizontal line, 4 ft. apart, is just straight and unstretched ; but when a certain weight is attached to the middle point, the string is stretched by two inches. If the modulus of elasticity of the string be equal to a weight of 300 lbs., find the weight attached, and the work done.

6. If l be the unstretched length of a string which is tied to two points in the same horizontal line at a distance l' from each other ($l' > l$), show that when a weight W is attached to the middle point, the inclination θ of each portion of the string to the vertical in the position of equilibrium, is given by the equation

$$\frac{W}{2E}\tan\theta + \sin\theta = \frac{l'}{l}$$

where E is the modulus of elasticity.

7. A uniform rod AB weighing 12 lbs., is hung from a rigid horizontal beam by three equal wires, one at each end and one at the middle point. A weight of 8 lbs. is attached to the rod at C where AC = $\frac{1}{3}$ AB. If the wires obey Hooke's Law, find the pull in each wire.

8. A uniform elastic string of length l would have its length doubled by a pull equal to its own-weight. Find its length when hanging vertically from one end.

9. An unstretched elastic string without any weight has n equal weights attached to it at equal distances and is then suspended from one end. Prove that the increase of length is half what it would be if the same string were stretched by a single weight equal to $n+1$ times each of the former weights hanging at one end.

10. A uniform heavy chain 10 ft. long is suspended from two points in the same horizontal plane 108 ft. asunder; show that the tension at the lowest point is approximately 1·477 times the weight of the chain.

11. The entire load of a suspenion bridge is 320 tons, the span is 640 ft., and the height is 50 ft. Find the tensions at the points of support and also the tension at the lowest point.

12. Prove that the work done in stretching an elastic string of natural length l and modulus λ, from tension T_1 to tension T_2, is

$$\frac{l}{2\lambda}\left(T_2^2-T_1^2\right)$$ [*P. U. 1937*]

13. Four equal smooth jointed rods AB, BC, CD, DE hang in equilibrium from two points A, E in the same horizontal level: the rods are light but carry weights W at their ends. Draw a force diagram, taking the horizontal component of the reaction at A and E to be W, Calculate the length of each rod when AE is 40 ft.

14. A flexible uniform light string is stretched over a plane curve; show that the whole pressure on any part of the curve varies as the angle between the tangents at the extremities of that part or as the difference between the tensions at the extremities according as the curve is smooth or rough.

15. If a light string is passed round and round a post or bollard, show that the tension is increased in the ratio $e^{2\mu\pi}$ for each complete turn.

16. A uniform string of weight w passes once round a horizontal cylinder of radius r, cutting the generators at right angles. Show that its length cannot be less than 13·58r.

17. The wire for a telegraph line cannot sustain a weight greater than one equivalent to a mile of its own length without breaking. If the wire is stretched on poles at equal intervals of 88 yards, what is the least sag permissible?

18. A uniform elastic string has a length a_1 when the tension is T_1 and a length a_2 when the tension is T_2. Show that its natural length is

$$\frac{(a_2 T_1 - a_1 T_2)}{T_1 - T_2}$$

and that the amount of work done in stretching it from its natural length to a length $(a_1 + a_2)$ is

$$\frac{1}{2} \frac{(a_1 T_1 - a_2 T_2)}{(T_1 - T_2)(a_1 - a_2)}$$ (*P. U. 1944.*)

REVISION PAPER

1. A heavy rough body rests on a plane of inclination α. Show that the least force acting down the plane, which will cause the body to move down is $W \sin(\lambda - \alpha) \sec \lambda$, where λ is the angle of friction and W is the weight of the body. (*P. U. 1935 Supp.*)

2. A uniform rod is in limiting equilibrium, one end resting on a rough horizontal plane and the other on an equally rough plane inclined at an angle λ to the horizon. If λ be the angle of friction, and the rod be in a vertical plane, show that the inclination θ of the rod to the horizon is given by

$$\tan \theta = \frac{\sin(\alpha - 2\lambda)}{2 \sin \lambda \sin(\alpha - \lambda)}.$$ (*P. U. 1936*)

3. A right cone is placed with its base on a rough inclined plane; if $\frac{1}{13}$ be the coefficient of friction, find the angle of the cone when it is on the point of both slipping and turning over. (*D. U. 1937*)

4. Two rough particles connected by a light string rest on an inclined plane. If their weights and corresponding co-efficients of friction are W_1, W_2, μ_1, μ_2 respectively, show that the greatest inclination of the plane for the equilibrium is

$$\tan^{-1} \frac{(\mu_1 W_1 + \mu_2 W_2)}{W_1 + W_2}.$$ (*P. U. 1940*)

5. A uniform ladder rests in limiting equilibrium with one end on a rough horizontal plane, and the other against a

smooth vertical wall; a man then ascends the ladder. Show that he cannot go more than half way up. (*P. U. 1935*)

6. A drawer is to be pulled out by a single force parallel to the drawer. Find how far from the middle of the end of the drawer the force can be applied so as not to cause the drawer to jam. (*P. U. 1938*)

7. A uniform straight rod, of length $2c$, is placed in a horizontal position as high as possible within a hollow rough sphere of radius a. Show that the line joining the middle point of the rod to the centre of the sphere makes with the vertical an angle $\tan^{-1}\dfrac{\mu a}{\sqrt{a^2-c^2}}$ where μ is the co-efficient of friction. (*P. U. 1937 Supp.*)

8. A rod rests in vertical plane within a hemispherical bowl, whose radius is equal to the length of the rod. If μ be the co-efficient of friction between the rod and the bowl, show that in the limiting equilibrium the rod makes an angle with the horizontal equal to $\tan^{-1}\dfrac{4\mu}{3-\mu^2}$.

(*P. U. 1931 (Hons.) B. A.*)

9. A cubical block of wood is placed on a rough horizontal table, and a horizontal force is applied at the middle of one of the sides of the square top at rt. angles to that side. Prove that as force is increased the block will begin to slide if the co-efficient of friction is less than $\frac{1}{2}$ and that otherwise it will begin to tilt. (*P. U. 1931*)

10. A full reel of cotton of radius a is kept at rest with the axis horizontal on a plane of inclination α, by holding the thread which comes off horizontally and tangentially from the middle of the reel. If the weight is W, find the tension of the thread and the friction between the reel and the plane. Find also the least co-efficient of friction for which the reel will rest in this way. (*P. U. 1941*)

11. Two rings of equal weight are free to move on a rough horizontal rod, the co-efficient of friction being μ. They are connected by a smooth string on which another of double the weight can move freely. Prove that the tension

in the string (when the system is in equilibrium with the two rings as far apart as possible) is W $\sqrt{1+4\mu^2}$. Find also the inclination of the string in this position. (*P. U. 1940 Supp.*)

12. A uniform rod of length $2a$ rests with one end on a rough plane of inclination α to the horizontal, and lies in a vertical plane through a line of greatest slope of the plane. It is supported by a smooth peg at a distance c from the end in contact with the plane. Show that if the end of the rod is about to slip up the plane and λ is the angle of friction, the inclination θ of the rod to the horizontal is given by

$$c \sin(\alpha+\lambda) = a \cos\theta \sin(\theta+\alpha+\lambda). \qquad (\textit{D. U. 1941.})$$

13. A ladder which stands on a horizontal ground, leaning against a vertical wall, is so loaded that its centre of gravity is at distances a and b from its lower and upper ends respectively. Show that if the ladder is in limiting equilibrium; its inclination θ to the horizontal is given by

$\tan\theta = \dfrac{a-b\mu\mu_1}{(a+b)\mu}$ where μ, μ_1 are the co-efficients of friction between the ladder and the ground and between the ladder and the wall. (*D. U. 1940*)

14. Prove that the work done in stretching an elastic string AB of natural length l and modulus λ from tension T_1 to the tension T_2 is $\dfrac{l}{2\lambda}(T_2^2-T_1^2)$. (*P. U. 1937*)

15. An elastic string has its ends attached to two points in the same horizontal plane and initially it is just tight and unstretched; a particle of weight W is tied to the middle point of the string; if the modulus of elasticity be $\dfrac{W}{\sqrt{3}}$ show that in the position of equilibrium the two portions of the string will be inclined at an angle of 60° to one another. (*P. U. 1941*)

16. An elastic string ABC, whose modulus of elasticity is 4 W, is tied at A to a fixed point. At B and C are attached weights each equal to W, the unstretched lengths of AB and BC being each equal to c. Show that, if the string and bodies take up a vertical position of equilibrium, the stretched

lengths of AB and BC are $\frac{3c}{2}$ and $\frac{5c}{4}$ respectively.

(*P. U. 1939*)

17. Show that the work done in expanding an elastic string of length l_0 and of elastic modulus λ to a length l is

$$\frac{\lambda(l-l_0)^2}{2l_0}.$$

A body of mass m rests on a rough inclined plane whose inclination α to the horizon is greater than β, the angle of friction. It is held at rest by an elastic string attached to it and to a point on the plane. It λ is the modulus of elasticity, show that the potential energy of the string in position of equilibrium is

$$\frac{l_0 m^2 g^2 \sin^2(\alpha-\beta)}{2\lambda \cos^2\beta}.$$

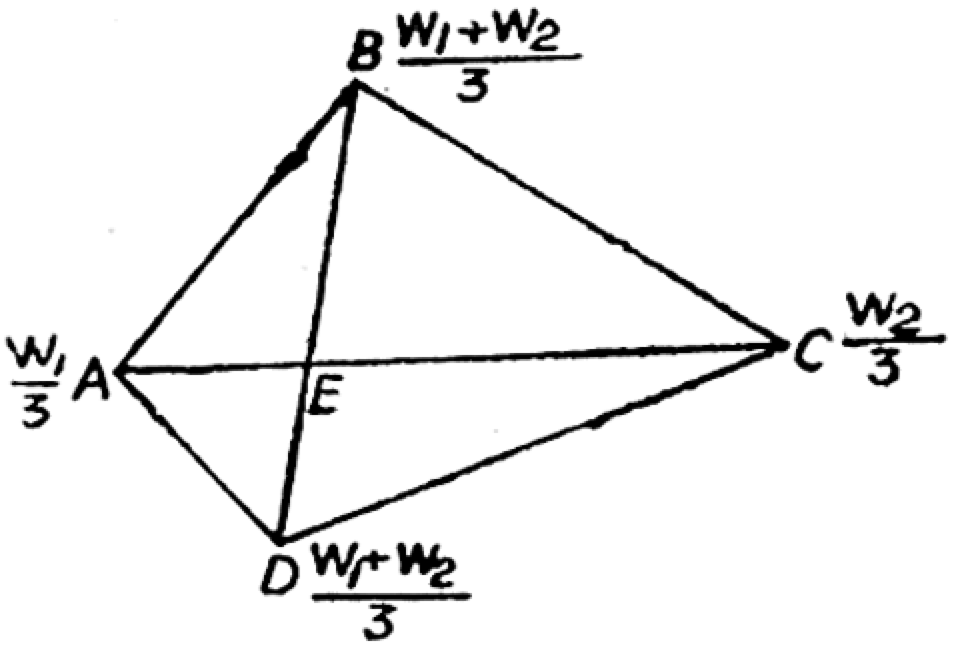
(*P. U. 1940*)

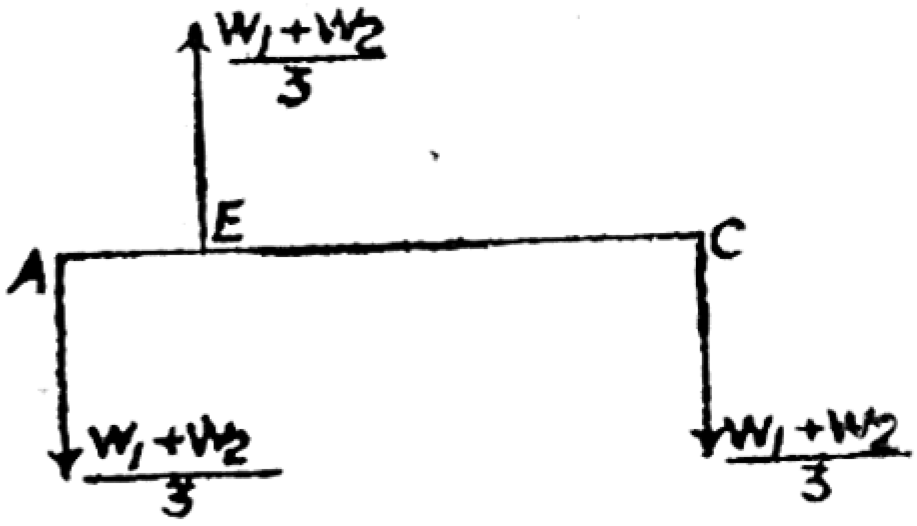
18. An elastic string of natural length l inches when used to pull a body of weight W lbs. steadily along a rough level table stretches $\frac{l}{10}$ inches. If the body is lifted vertically, the extension is $\frac{9l}{20}$ inches ; show that co-efficient of friction of the table is $\frac{2}{9}$.

(*P. U. 1940 Supp.*)

APPENDIX.

The centre of gravity of any quadrilateral area is the same as that of four equal particles placed at its angular points, together with another equal particle of negative mass placed at the intersection of the diagonals.





Let ABCD be the quadilateral lamina and let the diagonals AC, BD intersect at E.

Let W be the weight of the whole quadrilateral. If W_1, W_2 be the weights of △s ABD, BCD, then

$$\frac{W_1}{W_2}=\frac{AE}{EC} \qquad \text{......(1)}$$

Replace wt. of △ ABD by weights each equal to $\frac{W_1}{3}$ at A, B, D ; similarly replace wt. of △ BCD by equal particles each equal to $\frac{W_2}{3}$ at B,C,D.

We, thus, have weights

$\frac{W_1}{3}$ at A, $\frac{W_2}{3}$ at C, and $\frac{W_1+W_2}{3}$ at each of the points B and D.

Keeping the weights at B and D undisturbed, *we shall show that*

(*i*) the C.G. of 2 weights $\frac{W_1}{3}$ at A and $\frac{W_2}{3}$ at C is the same as

(*ii*) the C.G. of 3 weights $\frac{W_1+W_2}{3}$ at A $\frac{W_1+W_2}{3}$ at B and the negative weight $\frac{W_1+W_2}{3}$ at E. The sum of the weight in (*i*) and (*ii*) being the same.

Now EG_1 in case (*i*) taking E as origin is by Art. 83, given by

$$EG_1 = \frac{\frac{W_1}{3}\times(-AE)+\frac{W_2}{3}EC}{\frac{W_1}{3}+\frac{W_2}{3}} = \frac{\frac{W_1}{W_2}(-AE)+EC}{\frac{W_1}{W_2}+1}$$

$$= \frac{\frac{AE}{EC}\times(-AE)+EC}{\frac{AE}{EC}+1} = \frac{EC^2-AE^2}{EC+AE}$$

$$=EC-AE. \qquad \ldots\ldots(2)$$

Similarly

EG_3 in case (*ii*) is given by

$$EG_2=\frac{\frac{W_1+W_2}{3}(-AE)-\frac{W_1+W_2}{3}\times 0+\frac{W_1+W_2}{3}\times EC}{\left(\frac{W_1+W_2}{3}\right)-\left(\frac{W_1+W_2}{3}\right)+\frac{W_1+W_2}{3}}$$

$$=EC-AE \qquad \ldots\ldots(3)$$

showing that G_1 and G_2 are coincident.

Corollary. If from CA, we cut off CF=AF, then we deduce from (2) or (3) that F is the C. G. of weights

$\frac{W_1}{3}$ at A and $\frac{W_2}{3}$ at C.

$$EC-CF=EC-AE=EG_1.$$

ANSWERS

EXERCISES I (A), Page 15.

1. (*i*) $R=\sqrt{201}$. (*ii*) $\alpha=60°$. (*iii*) $\alpha=90°$. (*iv*) $\alpha=60°$. (*v*) $P=30+5\sqrt{13}$; $Q=30-5\sqrt{13}$. (*vi*) $\frac{P}{Q}=\frac{\sqrt{3}}{1}$. 2. $P=10\sqrt{3}$; $Q=10$. 3. $-\frac{5}{8}$. 4. $P=4\sqrt{3}$ lbs. 5. $\frac{5}{3}$. 6. $R=32$. 7. $\frac{\text{smaller force}}{\text{greater force}}=\frac{1}{\sqrt{2}}$. 8. 120°.

EXERCISES I (B), Pages 19—21.

1. $2\sqrt{37}$ lbs. wt., making angle $\tan^{-1}\frac{3\sqrt{3}}{11}$ with the first force : (*ii*) $2\sqrt{13}$ lbs. wt., making angle $\tan^{-1}\frac{3\sqrt{3}}{5}$ with first force. **2.** 7 lbs. wt., inclined at an angle $\tan^{-1}\frac{24}{7}$ with the resultant. **3.** $5\sqrt{2}$ lbs. wt., or $-10\sqrt{2}$ lbs. wt. The negative result $-10\sqrt{2}$ signifies, that if the second component be taken in the opposite direction, the resultant will be the same; in this case the forces of 5 lbs. wt. and $10\sqrt{2}$ lbs. wt. will act at 135°. **4.** Maximum, when forces act along the same direction and minimum when they act in opposite directions. **5.** 32·5 lbs. wt. and 7·5 lbs. wt. **8.** 6°. **11.** 5 and 12 lbs. wt. **12.** 4 and 8 lbs. wt. ; the resultant is doubld. **13.** $R\sqrt{2}$; 45°. **14.** R, $\frac{R}{2}(\sqrt{6}-\sqrt{2})$. **17.** $P\sqrt{2}(\sqrt{3}+1)$, where P is the magnitude of the equal forces. **18.** Each component is equal to $\frac{R(\sqrt{3}-1)}{2}$.

EXERCISES II, Pages 31—33.

4. 90°, $\sin^{-1}\frac{8}{17}$ $\sin^{-1}\frac{15}{17}$. **6.** (*i*) 5 and 7 lbs. act in the same direction which is opposite to that of 12 lbs., (*ii*) equilibrium is not possible, (*iii*) angle between first two

forces is 90° ; and between first and third is 120°. **7.** 25
lbs., $\frac{25}{2}(\sqrt{6}+\sqrt{2})$ lbs. **10.** Magnitude equal to any of the
equal forces and direction parallel to the third force. **11.**
$6\sqrt{3}$ lbs. along AD. **12.** Represented in magnitude and
direction by $\frac{19}{2}$ AD **13.** 6 lbs. parallel to the fifth force.
16. $\tan\theta=\frac{1}{2}$ $\therefore$ $\sin\theta=\frac{1}{\sqrt{5}}$ $\cos\theta=\frac{2}{\sqrt{5}}$.

EXERCISES III, Pages 37—38.

1. Centroid of the triangle. **5.** Centroid of the quadrilateral. **9.** $\theta\sqrt{3}$.

EXERCISES IV, Pages 42—44.

1. 120° ; 90° and 30°. **2.** 8 and 15. **3.** $P:Q::\sqrt{2}:\sqrt{3}$.
6. 45° ; 60° ; ratio of the resultant to the second
force$=(\sqrt{3}+1)/\sqrt{2}$. **10.** The triangle must be equilateral.

EXERCISES V, Pages 49—52.

1. Horizontal force$=10\sqrt{3}$ lbs. wt. ; tension $-$ 20 lbs.
wt. **2.** Horizontal force$=W(2+\sqrt{3})$; tension $=W\sqrt{2}(\sqrt{3}+1)$.
3. $W\sqrt{6}(\sqrt{3}-1)2$, $W(\sqrt{3}-1)$. **4.** 12 : 5 ; 12 W/13
lbs. wt., where W is the given weight. **5.** 68 lbs. wt., 32
lbs. wt. **6.** 5/6 of a lb. wt. **7.** The forces lie on the
opposite sides of the normal, in the same vertical plane.
Reaction$=2$ lbs. wt. **8.** $Q=P\sqrt{3}$, $P\leqslant\frac{1}{2}Mg$. **9.** Pressures on the nail and the plane are $\frac{5}{2}$ lbs. wt. and $\frac{5\sqrt{3}}{2}$ lbs. wt.
10. $W\tan\alpha$. **11.** Force and Reaction are each
equal to 57·7 lbs. wt. approximately. **12.** If each weight
be W, reaction$=1{\cdot}414\,W$, and it makes angle of 45° with
the horizon. **13.** Reaction at $B=W\sqrt{3}$; at A and C, each
$=\frac{\sqrt{3}-1}{\sqrt{2}}W$, where W is the weight. **14.** W and $W\sqrt{3}$ at
the lower and upper pegs, respectively. **15.** $W(\sqrt{3}+1)/\sqrt{6}$
lbs. wt. **17.** $W\sqrt{2}$, W. **18.** $2W\cos\theta$, W. Tension in
each of the inclined portions of the string$=Wl/d$, and in the
horizontal portion$=(c-l)\,W/2d$, where $d^2=l^2-(c-l)^2/4$. **22.**

25 lbs wt. **23.** 45 lbs. wt. **24.** Tension in DA=Tension in BE=2lbs., tension in AB=tension in BC=tension in CA

$$= \frac{2\sqrt{8}}{3} \text{ lbs.}$$

EXERCISES VI, Pages 59—62.

1. (*i*) 6 lbs. at distances of 8″ and 4′ from the 1st and the second force, respectively. (*ii*) 12 lbs. at distances of 1·2″ and 10″, (*iii*) 24 lbs. at $10\frac{5}{16}$ and $7\frac{11}{16}$ inches, (*iv*) 80 grammes at $10\frac{15}{16}$ and $14\frac{1}{16}$ cms. **2.** (*i*) 2 lbs., in the direction gf the greater force at 2′ and 1′, (*ii*) 2 lbs. at 7′ and 5′, (*iii*) 3·5 lbs. at $5\frac{5}{28}$ and $4\frac{11}{28}$ ft., (*iv*) 10 gms. at ·875 and 1·125 metres.

3. $3\frac{3}{4}$ lbs. wt.; $8\frac{3}{4}$ lbs. wt. **4.** Pressure on A=42 lbs. wt. and on B=49 lbs. wt. **5.** $1\frac{13}{15}$ ft. from the end. **6.** 32 lbs. **7.** P=27 lbs. wt., Q=18 lbs. wt. **8.** 2 : 3. **9.** 4·5 kilogm. **10.** $bl/(a-b)$, where $2l$ is the length of the girder. **11.** $1\frac{1}{3}$ ft. **17.** The resultant cuts BC and BA produced at distances equal to $\frac{7}{2}a$ from B when a is a side of the square. **19.** $16\frac{1}{2}$ inches; $13\frac{1}{2}$ inches. **20.** 45 lbs. wt.

21. $\frac{mP}{m+n}$; $\frac{nP}{m+n}$, $\frac{ma}{n}$. **22.** 2 ft. from the end about which it can turn.

EXERCISES VII, Pages 71—73.

1. $3\frac{1}{9}$ inches from the 6 lbs. wt. **2.** 6. **3.** 36, $2\frac{1}{2}$ ft. from 12 lbs. wt. **4.** 3 inches from the force 8.

5. $\frac{W_1}{2}\;\frac{(W_1+W_2-2W_0)}{W_0-W_2}$. **7.** $1\frac{11}{15}$ feet from the end. **8.** 0, 380. **9.** 5, inclined at $\tan^{-1}\frac{3}{4}$ to AC. **10.** $26\frac{2}{3}$ lbs. **11.** $5\sqrt{2}$, it cuts BA produced at X where AX=$\frac{4}{5}$ ft. and is ∥ to or AC. **12.** $P\sqrt{8}$ perpendicular to BC and cuts BC at Q where CQ=$\frac{3}{2}$ BC. **14.** $\frac{1}{2}a\sqrt{2}$ from the base of the tree. **15.** $2\sqrt{3}$, inclined at 30° to BC and cuts BC at $\frac{8}{2}$ BC from C. **16.** $2\sqrt{5}$ inclined at an angle 210° to AB, it cuts

AD produced at X, where DX = 2. **20.** 40 lbs. **21.** $33\frac{1}{3}$ lbs. **22.** $1\frac{8}{17}$ ft. **23.** Not nearer than 7 inches to either string.

EXERCISES VII A, Pages 74—76.

1. $10\frac{1}{6}$ lbs. wt. and $7\frac{5}{6}$ lbs. wt. **2.** A wt. of 4 lbs., attached at $1\frac{1}{2}$ ft. from B. **3.** Weight = $12\frac{1}{2}$ lbs. In the first case, pressure at C = 25 lbs. wt.; in the second case pressure at D = $37\frac{1}{2}$ lbs. wt. **4.** $3\frac{1}{2}$ ft. **6** A force of 2 lbs. acting in direction DA at a distance equal to $\frac{3}{2}$ AB from A. **7.** $4\sqrt{2}$ lbs. parallel to AC, and cutting BA produced at a distance of $\frac{3}{2}$ AB from A. **8.** 9 lbs., wt.; divides AD in ratio 1 : 2. **9.** Resultant F is given by $F^2 = P^2 + Q^2 + R^2 - \sqrt{2}.\ R(P+Q)$: the angle which it makes with AB is $\tan^{-1}\left(-\frac{Q - R\sqrt{2}}{Q - P\sqrt{2}}\right)$, and it cuts BA produced at X such that $AX = Qa/(R\sqrt{2} - Q)$. **10.** 15 lbs. force through the centre is $4\sqrt{26}$ lbs. and makes angle θ with AB, such that $\cos\theta = \frac{1}{\sqrt{26}}$, and $\sin\theta = -\frac{5}{\sqrt{26}}$. **11.** Pressures exerted by the front and back wheels are 10 st. $11\frac{1}{3}$ lbs. and 2 st. $6\frac{2}{3}$ lbs., respectively. **12** The resultant is $2\sqrt{3}$ lbs. wt., inclined at angle $-150°$ with AB, and meeting BA produced at X, where $AX = \frac{5a}{2}$.

EXERCISES VII-B, Pages 81—83.

2. The centroid of the quadrilateral, *i.e.*, the midpoint of the line joining the mid points of the opposite sides. **3.** $2\frac{6}{7}$ in. **12.** $\frac{105\sqrt{2}}{2}$ lbs. wt. **14** $P = 10\sqrt{2}$ lbs.: $Q = 20\sqrt{2}$ lbs., and acts along DB.

REVISION PAPER I, Pages 83—84.

1. $8\sqrt{3}$, $16\sqrt{3}$ lbs. **3.** W sec θ, W tan θ.

12. $\frac{W(l-a)+Pa}{W}$.

EXERCISES VIII, Page 92.

2. $P = 10$ lbs., $Q = -35$ lbs.; moment $= 20$ ft. pounds. 3. each force $= 6\sqrt{3} - 4$ lbs. wt.

EXERCISES VIII A, Page 97—98.

3. $2\lambda\Delta$, where Δ is the area of the triangle. 4. $P = -10$ lbs., $Q = 60$ lbs.

EXERCISES IX, Pages 106—109.

1. 6 and $2\sqrt{73}$ lbs. wt., respectively. 2. Tension $= \frac{1}{8}\sqrt{2}$ cwt, force at the hinge $= \frac{1}{8}\sqrt{2}$ cwt. inclined at 45° to the rod. 3. 30° with the vertical; 40 lbs. 4. The vertical line passes through the peg and the middle point of the rod. Tensions are 7·2 lbs. and 9·6 lbs. 5. 50 lbs. wt.; inclined at $\cos^{-1}\frac{11}{14}$ to the vertical. 6. $10\sqrt{3}$ and $10\sqrt{7}$ lbs. wt. 7. 10 and $10\sqrt{3}$ lbs. 8. $3\sqrt{3}$ lbs., 3 lbs. 12. The supports are in a horizontal line and the weight $= 120$ lbs.

13. $\frac{2}{3}\sqrt{3}$ W. 21. $T = \frac{2W\sqrt{3}}{3}$; $R = W\frac{\sqrt{3}}{3}$.

EXERCISES X, Pages 119—123.

4. 20 lbs. wt. 5. $40(\sqrt{2}+1)$ lbs. wt. 6. On the line joining the centre to the leg opposite the missing leg, and at a distance from the centre equal to one-third of the diagonal of the square. 12. Thrust in BD $= 10$ kilog. Tensions in AB, CD, each $= \frac{20}{13}\sqrt{13}$ kilog. Tensions in BC, AD, each $= \frac{20}{13}\sqrt{13}$ kilog. 14. 125 lbs., 25 lbs, 75 lbs.

20. Thrust in BC $= \frac{1}{\sqrt{3}}$ W; Tension in AB or AC $= \frac{2}{\sqrt{3}}$ W, and tension at A $= 2$W, where W is each weight. 23. $T = 3W$; $R = \frac{1}{2\sqrt{3}}W$, $S = \frac{\sqrt{13}}{2\sqrt{3}}W$ where W is the weight of each bar. 26. $\frac{3W\sqrt{3}}{2}$ along BF; $\frac{W\sqrt{3}}{2}$ along CE.

27. $\frac{8}{18}$ oz. is the weight which the tradesman puts on to the weight-pan to show that the balance is true. He makes an extra profit of $\frac{1}{14}$ oz. per oz. of goods sold. 29. (*i*) At upper

hinge, $\frac{3}{4}$ W and horizontal (*ii*) lower hinge $\frac{5}{4}$W at angle 36° 35′ with the vertical.

REVISION PAPER II, Pages 123—127.

3. $2\sqrt{3}$ lbs. wt. inclined at 150° to AB ; moment of the couple $3a$ units of moment, where a is the in-radius. Resulant meets AB at P where BP = AB.

8. 120 lbs. 9. $l.\,\frac{P+Q}{Q}$. 13. $\cot^{-1}\frac{(\cot\beta-\cot\alpha)}{2}$.

20. $\frac{a}{l}$ must be <1 and $>\frac{1}{2}$.

21. $\tan^{-1}\left[\frac{a-b}{a+b}\tan\sin^{-1}\frac{W}{2P}.\right]$

EXERCISES XI, Pages 136—188.

4. One sixth of the hypotenuse. 7. $1\frac{1}{3}$. 1, $\frac{4}{5}$, ft. 13. $\tan^{-1}\frac{5}{24}$. 14. $6\frac{1}{4}$, $4\frac{1}{2}$, or $4\frac{1}{4}$ lbs. wt. at the corner opposite to the side 5, 12, or 13 ft.

EXERCISES XI A, Pages 150—154.

1. $\frac{a\sqrt{5}}{4}$. 2. $\sqrt{5}$ in. 3. $\frac{a}{4\sqrt{3}}$. 4. 4 in. 5. 4 in from A. 6. $2\frac{6}{9}$ ft. 7. 2·5 ft. 8. 5·6 in. ; 4 in. 9. 21·9 in. nearly. 11. 4 in. towards the 5 lbs. wt. on the line joining this weight to the 2 lbs. weight. 12. 8·4 ft., 3 ft. 15. $\frac{a}{\sqrt{5}}$. 16. $\frac{4a}{7}$.

22. If the masses be joined to form a triangle, they would be proportional to the sines of the angles subtended at the centre by the opposite side. For the problem to be possible, any two masses should be in the opposite sides of the diameter through the third. 23. $\sin^{-1}\frac{24}{25}$. 24. $\sin^{-1}\frac{336}{625}$. 25. At $\frac{1}{9}$ AB from O towards the mid-point of CD. 26. On the axis of symmetry at a distance $\frac{3}{26}(4-\sqrt{3})a$ from the base of the triangle. 27. $\frac{8-\pi}{6-\pi}$. 1. 28. (*i*) P : Q ; R : : $a : b : c$, (*ii*) P : Q : R $= -a : b : c$, or $a : -b : c$, or $a : b : -c$. (*iii*) P : Q : R = sin 2A : sin 2B : sin 2C. (*iv*), P = Q = R. (*v*) P : Q : R = tan A : tan B : tan C.

30. If O be the centre of the square, G and G′ the C.G. in the two cases, respectively, VG$=\frac{1}{21}$ AC. OG′$=\frac{1}{48}$. 33. $2a+b : 2b+a$. 37. $OG_2=\frac{1}{6}r$.

40. Centre of the whole must be 16 inches from the centre of the disc.

41. A′ bisects the median AD.

42. $\frac{11}{18}$ inch from the centre of the square.

EXERCISES XII, Pages 165—168

2. On the axis at distance $1\frac{80}{249}$ ft. from the top. 3. At one-third of the diagonal from the point where the weight is suspended. 7. The height of the part scooped out is one-third the height of the cone. 9. If $AB=2r$. $AP=2a$, $BP=2b$, the C.G. is

$$\left(\frac{r(a^2-b^2)-(a^3-b^3)}{r^2-a^2-b^2},\ \frac{\pi}{4}\ \frac{r^3-a^3-b^3}{r^2-a^2-b^2}\right)$$ where OB and a perp. through O are the axes. 10. (P+Q b/Q.

EXERCISE XIII, Pages 182—184

7. (*i*) Yes, (*ii*) unstable, (*iii*) $\frac{2}{3}$ of a lb. 8. When inclination of the plane$=\frac{\pi}{4}$. 9. Stable or unstable, according as $r>$, or $<h\sqrt{2}$, where h is the height of the cylinder and r the radius of the base. 18. 3·66 lbs. 20. $\cos\alpha=\frac{3}{5}$, where α is the semi-vertical angle. 24. $\frac{\pi h}{3}(R^2+Rr+r^2)$, where r, R are the radii, and h the altitude.

26. $$\frac{1}{2}\frac{a^2+ab+b^2}{a+b};\ \frac{2}{3\pi}\ \frac{a^2+ab+b^2}{a+b},$$ where a, b are the diameters. The point of contact is taken as the origin and the common diameter as x-axis. 27. At the centre in each case. 28. At distances $\frac{a}{2}$, $\frac{3}{8}$ a respeetively, from the centre on the axis of symmetry.

REVISION PAPER III, Pages 184—186.

2. 6 inches sq. is cut out from corner C of rect. ABCD', taking AB, AD as axes of x, y C.G. is given by

$x=\frac{85}{11}$ in. $y=\frac{37}{11}$ in. **8.** $OG=2\frac{6}{7}$ in.

10. $OG=\frac{cr^3}{R^3-r^3}$, G being in the prolongation of line joining centres O′,O. **12.** $\frac{4a}{3\pi}$.

EXERCISES XIV, (Work) Pages 193—194.

1. $217\frac{1}{7}$ ft. tons : ·24 H. P. **2.** 3·36 H. P. **3.** $21\frac{34}{35}$ ft. tons. **4.** $1064\frac{4}{5}$ ft. tons. **5.** $2109\frac{3}{8}$ ft.-tons ; $2\frac{17}{44}$ hrs. **6.** 1·4 H. P. **7.** $234\frac{3}{8}$ lbs. wt. **8.** 17·6 H. P. **9.** H is given by $H\times 550=m\times 2240\times\frac{2v}{15}\times\frac{1}{n}+mx.\frac{22v}{15}$.

10. 25·7 *m. p. h.* approximately.

EXERCISES XV (Virtual Work). Pages 207—209.

2. 2W. **8.** $\frac{3\sqrt{3}}{2}$ W.

EXERCISES XVI, Pages 224—225

1. 10 lbs. **2.** $6\sqrt{3}$ lbs. **3.** $22\frac{1}{2}$ lbs., $\frac{45\sqrt{3}}{2}$ lbs.

4. $\frac{1}{4}$. **5.** $\frac{1}{\sqrt{3}}$ **7.** $\frac{W\sin(\alpha+\lambda)}{\cos(\theta-\lambda)}$; $\frac{W\sin(\alpha-\lambda)}{\cos(\theta+\lambda)}$.

9. $\tan^{-1}\left(\mu.\frac{W_1+W_2}{W_1-W_2}\right)$. **10.** $\sin\alpha=\sin\beta-\mu\cos\beta$.

EXERCISES XVII, Pages 234—240.

1. $\tan^{-1}\frac{18}{17}$. **2.** $\tan^{-1}(2\mu)$. **3.** $\frac{1}{\sqrt{3}}$. **4.** 2.

5. Half the length of the ladder ; $\tan^{-1}\frac{a-b\mu\mu'}{\mu(a+b)}$.

8. $\cos^{-1}\frac{PW}{Q\sqrt{P^2+W^2}}$. **9.** $\sin^{-1}\frac{Q-P}{2\sqrt{W^2-PQ}}$

10. $W\frac{2\mu-\tan\alpha}{\tan\alpha-\mu}$. **13.** 0,2 $\tan^{-1}\frac{1}{2}$; **14.** $\cos^{-1}\frac{\sin\alpha-\mu}{1-\cos\alpha}$. **15.** Unity. **16.** Slide. **17.** $\tan^{-1}\frac{1}{2}$; height=four times the radius. **18.** Topple if $\mu>\frac{a}{h}$, h being the height and a the length of the edge which is not horizontal.

20. $\tan\ \theta=\frac{\mu(P+3Q)}{P+Q}$, θ being the inclination of AC to the vertical. **22.** $\frac{24}{43}$ **27.** $\tan^{-1}\frac{P\mu+Q\mu'}{P+Q}$. **28.** Angle of friction.

EXERCISES XVIII, Pages 251—258.

1. 2″, $\frac{5}{6}$ ft. lbs. **3.** $\frac{3}{4}$ ft. lbs. **4.** 288 ; 49. **5.** 7 lbs. ; $4\frac{11}{12}$ ft. lbs, **8.** $\frac{3}{2}l$. **12.** 13·8 ft.

REVISION PAPER IV, Pages 253—256.

3. $2\tan^{-1}\frac{1}{52}$. **6.** See Ex. 8 worked out P. 233.

10. $T=\frac{W}{2}\sin\alpha$; least $\mu=\frac{1}{2}\tan\alpha$.

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